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ABSTRACT

This book is an outgrowth of a 2-year study of 284 children discovered during preschool or kindergarten to be advanced in mathematics. In addition to psychometric and cognitive testing conducted at the beginning, middle, and end of the study, half of the children attended biweekly interventions designed to enrich their experience with mathematics. Results found the children remained advanced in math over the 2-year period, their spatial reasoning related more closely to their math reasoning than did their verbal reasoning, and the math scores of the boys started and remained higher than those of the girls. The intervention proved effective in enhancing mathematical reasoning. The book discusses ways of identifying very young math-advanced children as well as a variety of educational strategies to meet their needs. Its primary emphasis is on creating an open-ended approach to teaching mathematics that provides an opportunity for children at different levels of advancement and different personal styles to engage with mathematical challenges in a playful way, to conceptualize math broadly, to pose problems, and to make sense of the mathematical system. Appendices include a questionnaire for parents, jobcards and other activities, annotated bibliography, and recommendations for the classroom. (Contains 84 references.) (CR)

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Teachers Nurturing Math-Talented Young Children

Barbara Waxman
Nancy M. Robinson
Swapna Mukhopadhyay
University of Washington
Seattle, Washington

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Barbara Waxman
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THE NATIONAL RESEARCH CENTER ON THE GIFTED AND TALENTED

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By far, our biggest debt goes to the participating children and their parents. We learned much from observing and teaching these lively, spirited children. They taught us, too.

Teachers Nurturing Math-Talented Young Children

Barbara Waxman
Nancy M. Robinson
Swapna Mukhopadhyay
University of Washington
Seattle, Washington

ABSTRACT

Talent in mathematical reasoning is highly valued in this society and yet very little is known about its early course. This book is an outgrowth of a two-year study of children discovered during preschool or kindergarten to be advanced in their thinking about math. In addition to psychometric and cognitive testing conducted at the beginning, middle, and end of the study, half of the children were randomly assigned to biweekly intervention (Saturday Club) for a total of 28 weeks over the two years. Among other findings, the study revealed that, as a group, the children remained advanced in math over the two-year period, that their spatial reasoning related more closely to their math reasoning than did their verbal reasoning (although they were ahead in all three domains), and that the math scores of the boys started and remained somewhat higher than those of the girls. The Saturday Club intervention proved effective in enhancing mathematical reasoning.

This book discusses ways of identifying very young math-advanced children as well as a variety of educational strategies to meet their needs. Its primary emphasis is on creating an open-ended approach to teaching mathematics that provides an opportunity for children at different levels of advancement and different personal styles to engage with mathematical challenges in a playful way, to conceptualize math broadly, to pose problems, and to make sense of the mathematical system. Also emphasized are the importance of representing and communicating mathematical ideas in multiple ways in order to deepen children's understanding. A variety of engaging activities such as the Fibonacci series, the Vedic square, and chip-trading are described. Most of these activities emanate from "big ideas" such as the nature of numerals and the number system, equivalence, visualizing and graphing numbers, measurement, estimation, and so on. Job cards for various mathematical tasks are included, as well as ways to integrate mathematics into other aspects of the curriculum. The approach to mathematics portrayed in this book is one that creative teachers can flexibly adapt to meet the needs of math-advanced children in a regular or specialized classroom.

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Teachers Nurturing Math-Talented Young Children

Barbara Waxman
Nancy M. Robinson
Swapna Mukhopadhyay
University of Washington
Seattle, Washington

CHAPTER 1: Young Children Advanced in Mathematical Thinking—How Can We Recognize Them?

This book is an outgrowth of a two-year study of children discovered during preschool or kindergarten to be advanced in their thinking about math. With the support of a Javits grant from the Office of Educational Research and Improvement (OERI) of the United States Department of Education, we studied 284 children over a period of two years and involved half of them in biweekly Saturday Clubs designed to enrich their experience with mathematics. The study, which we named "Math Trek," will be described in greater detail in Chapter 2. It furnished us with the basis for this book, which constitutes a description and elaboration of our experience with a wonderful group of lively young math-talented children in primary grades (K-1 or 1-2). How varied and enchanting a group they were!

This book is about teaching young children who are good at math, children ranging from those who are "pretty good" at math (perhaps in the top 10%) to those who are so remarkably advanced that they seem light-years ahead of their agemates. The children differ not only in their degree of advancement, but in many other ways as well. Some are better at verbal reasoning and/or visual-spatial reasoning than others—in fact, some are more advanced in one or more of those domains than they are in math. In some children, advancement in mathematical reasoning is part of a picture of high general intellectual capability, while in others, logical-mathematical "intelligence" (Gardner, 1983) seems more specialized.

Some of the children are deeply passionate about numbers, as is evident in their questions, in their tendency to ignore what the rest of the class is doing while they are absorbed with a problem of their own, and in their smiles of satisfaction when they make sense of something puzzling. They love "hard problems" and real challenges, and they are confident enough to try tasks they may not be able to master the first time. Other young math-advanced children, despite knowing a good deal about math, have picked up the negative math-attitudes of their parents, previous teachers, or older siblings. Others, by virtue of having been over-drilled in number facts and procedures, have already come to think of mathematics in negative ways. Such children communicate, by their behavior, that math is "boring." They don't have much fun with numbers; they don't experiment; lacking confidence, they tend to prefer easy problems to hard ones—or math-related activities that don't look like math and definitely are not labeled "math."

Add to this picture the fact that the contexts of these children's lives differ at least as widely as the children do. In some families, numbers are a valued and embedded part of living. Children from these families have been introduced to numerical concepts such as counting and size from their earliest conversations and their parents habitually involve them in number-thinking. Some families regard math as very important but introduce their children too early and too vigorously to written algorithms and number facts without the concepts or the playfulness that enable young children to "own" what they are learning. In other homes, parents, who inevitably encounter numbers every day of their lives, simply don't share these with their children. Recognize also that it is probably easier to learn about math if your family life is relatively ordered and reasonably predictable. Some families live more patterned lives than others; some families talk more than others; some families are barely surviving.

Now, add the widely varying contexts provided by schools. School districts differ in size, mixes of students, curriculum choices, political climates (specifically, politics about gifted children), and, above all, the degrees of flexibility encouraged by district and school administrators.

And, to this mix add teachers. Let's face it: some teachers love math and it is so much a part of their lives, their thinking, and their classrooms that children's notions of numbers, numerical relationships, and uses can't help but thrive. Some teachers don't know much about mathematical concepts or children's developmental trajectories in this area and, therefore, feel inadequate to stray beyond the prescribed textbook and materials. Sadly, far too many teachers were taught (as children) that math constituted an encapsulated area of study with esoteric rules and symbols to be memorized but not understood. Math was not supposed to be any fun at all. Those teachers decided that one could successfully live without it.

They were wrong, of course! Math is not only fun but reasonable; sense-making in math is at least as much fun as one can have anywhere else; mathematical symbols are no harder to decipher than any other language; and almost every teacher already knows plenty about math to teach primary-age children, even bright ones! Being stumped by a math-related question can be a nice occasion to join the children in experimenting with pathways to finding answers. It isn't scary; it's a blast! So the issue becomes how to transform math from a scary school subject to an engrossing, expanding aspect of teaching that is enjoyable and challenging for children and teacher alike. This book, drawing from our experience, represents an effort to describe such a transformative practice.

Some Common Characteristics of Math-Advanced Children

Keeping in mind that diversity abounds within the group we are talking about, here are some characteristics that many math-advanced children exhibit. No single child is likely to show them all.

- Advanced computational skills (not necessarily advanced problem-solving skills)
- Advanced problem-solving skills (not necessarily advanced computational skills)
- Rapidity of mastering typical math curriculum at an earlier age than classmates
- Exceptional mathematical reasoning ability and memory
- Interest in mathematical symbols and written representations
- Ability to hold problems in mind that aren't yet figured out—to ponder them from time to time until the answer emerges
- "Number sense"—a ballpark "feel" for whether an answer is reasonable or whether a procedure might be appropriate
- Frequent step-skipping in problem solving and unexpected ways of solving problems; capacity for inventing strategies
- Rapid and intuitive understanding—thinking faster than they can write their answers or describe their procedures

- A tendency to choose mathematics when presented with a choice of activities
- Awareness of numbers in their surroundings and a tendency to frame questions numerically ("How many minutes to recess?")
- Interest in looking for patterns and relationships and explaining them
- Willingness and capability for doing problems abstractly; often preferring not to use concrete aids or manipulatives that are the hallmark of current approaches to math education
- Conversely, fluency in representing mathematical ideas in different media—e.g., manipulatives, drawings, equations, graphs, stories
- From these translated representations, gaining new insights that other children don't see
- Long periods of absorption with problems in which they are truly engaged; reluctance to give up on an unsolved problem
- Treating road-blocks as challenges; detouring rather than retreating in the face of obstacles; "courageousness" in trying new pathways of thinking
- Propensity for seeing connections between a new problem and problems previously solved or ideas from an entirely different domain
- Pleasure in posing original, difficult problems
- Joy in working with "big" numbers
- Capacity for independent, self-directed activities
- Enjoyment of challenging mathematical puzzles and games

(Adapted from House, 1987, pp. 51-52, and expanded by ideas from Math Trek workshop participants)

These, then, are the math-talented children of Math Trek and of classrooms everywhere. No two are just alike, though they share the potential for advanced thinking about numbers, patterns, and connections. They can be very exciting and very rewarding students to teach, but they constitute a special responsibility for teachers. Unless they are challenged, given room to grow and incentives to do so, their talent can be seriously endangered. They can go only so far with inventing an appropriate math curriculum for themselves. Boredom and repeated experiences with "problems" well below their level of mastery can erode their joy and passion for exploring the system, their sense of efficacy as mathematicians, indeed, their basic mathematical reasoning abilities themselves.

Math Trek was undertaken to explore the development of young, math-talented children and to experiment with reasonable ways to extend the curriculum of the primary grades. In the next chapter, we will describe our Math Trek study. Chapter 3 describes a smorgasbord of alternative arrangements within and between classrooms, ways in which children can be exposed to challenging curricula that are optimally matched to their readiness and their individual sets of needs. In the remaining chapters, we will describe ways in which teachers can extend their usual classroom curricula by offering all their students open-ended activities that each one can profitably experience and re-experience at his or her own level. Finally, in the appendices, readers will find descriptions of specific activities as well as lists of additional books about young children and mathematics resources suitable for adapting in the ways we will describe.

CHAPTER 2: Math Trek—Identifying and Nurturing Mathematical Precocity in Young Children

The early discovery of talent is the first step toward nurturing and enhancing it. There are surely many children with special talents whose abilities are doomed to wither for lack of attention and encouragement (Feldman, 1986). Indeed, in the case of mathematical talent, advanced ability can be actively discouraged if children are forced to repeat, over and over, low-level skills they have mastered long before. As we have just seen, math-advanced children show their abilities in a variety of ways. It is up to the adults who nurture them—teachers and parents—to identify and engage the children in expanding their interests and competencies if their advancement is not to be lost.

Talent in mathematical reasoning is highly valued in this society and is basic to many career paths, especially those leading to science and technology. Yet, little is known about the very early course of mathematical talent, and we make no systematic effort to identify very young children who are talented at math. For older students who reason well mathematically, annual regional talent searches (Stanley, 1990) now involve some 160,000 or more seventh-graders who test their skills with high-level academic aptitude measures, namely, the *Scholastic Aptitude Test* (*SAT*) and the *American College Test* (*ACT*). We use these out-of-level tests meant for considerably older students (in this case, juniors and seniors applying for college) because seventh-graders who are math talented easily top out on measures standardized for their own age group. Similarly, using out-of-level tests standardized for eighth-graders, regional talent searches have recently been instituted for upper elementary school students. There are math contests of various kinds for junior and senior high school students and occasional contests for upper-elementary school students. But math-talented children in the primary grades have received no such attention and no concerted efforts have been directed at identifying them. Math Trek was directed toward finding out more about them.

Research About Very Young Math-Talented Students

Most of the research on math-talented students has also focused on the teenage years. The research has been possible in large part because investigators have had access to the participants in the talent searches mentioned above. Only a few investigators (Assouline & Lupkowski, 1992; Lupkowski-Shoplik, Sayler, & Assouline, 1993; Mills, Ablard, & Stumpf, 1993) have looked at older elementary students who are good at math, and no one has specifically looked at math-advanced children as young as the early primary grades.

Yet, learning about numbers begins well before children enter school. It is a process embedded both in number-related social activities with parents and peers (Saxe, Guberman, & Gearhart, 1987) and in the children's self-directed play. Many children play with objects that are structured to have regular relationships to one another, such as blocks of different sizes, train track pieces, Legos or Duplos, or simply groups of small objects such as cars, bugs, or crayons (Ginsburg, 1989). They talk about quantities with their parents ("more," "all gone," and "big girl" are some of the first words they learn). "Are we there yet?" leads to all kinds of conversations about time and distance. Cooking, going to the store, reading books, setting the table, even finding the right channel on the TV are all number activities that come naturally at home. Preschoolers know a great deal about numbers even if they do not fully grasp the concepts (Gelman & Gallistel, 1978).

About the time children enter school, the "5- to 7-year shift" occurs, a benchmark era during which children's thinking normally becomes systematic and begins to detach from its dependence on a specific context (Sameroff & McDonough, 1994). Very young children may have good mastery of an idea in one context (e.g., don't write on the walls of the living room) but not in another (nor in your bedroom either!). As children become more systematic and begin to apply their knowledge across contexts, they begin to take off in their mathematical reasoning and to grasp the notion that numbers have reliable, systematic relationships to one another and mean the same thing wherever you find them.

Hardly any investigators have looked at differences in children's rate of development in mathematical reasoning, certainly not at the ages in which we are interested. Although most research on the development of early mathematical thinking has ignored differences among children, some research has looked at distinctive characteristics in strategy use (see Geary & Brown, 1991; Siegler, 1995).

When children first begin to use numbers, they have to figure things out every time. Over and over, they may add two and three on their fingers to get five. Most children continue to engage in slow, effortful and often inaccurate computation during the primary grades using a variety of problem-solving strategies before they eventually shift to retrieving facts they know rather than figuring them out each time (Siegler, 1991). Even by first grade, one can see some individual differences in children's strategies (Siegler, 1991, 1995). Although most children do not show habitual automaticity—that is, apparently effortless recall of basic number facts—until fourth grade (Kaye, de Winstanley, Chen, & Bonnefil, 1989), gifted children usually begin to use retrieval, or automatic recall, sooner and more frequently than do average children of their age (Geary & Brown, 1991).

Even among children who are good at math, personality differences can play a role in the way they deal with situations. In fact, Siegler (1988) described three types of first-graders: good students, not-so-good students, and perfectionists. The perfectionists, who wanted to be sure to be right, performed as well as did the good students, had higher standards, and did much more checking than the other two groups, even though they could retrieve number facts from memory as well as the good students could.

Research Questions

The study from which this volume grew had a number of facets. So little was known about young children who are good at math that there were many questions to answer. Among them:

1. How can we identify young children who are good at math? Can parents pick them out? Will parents and test scores agree?
2. If young children are good at math, at what else do they excel? At these early ages, what kinds of cognitive abilities go along with the ability to reason well in the quantitative domain?
3. Are there gender differences in math precocity at this young age or do they appear later? What does entering school do to any gender differences that are observed?
4. How stable is mathematical precocity? Will young children who are initially good at math retain their rapid pace of development in mathematical reasoning?
5. And finally, if children are given extra experience with mathematical thinking in a friendly and engaging environment that invites inquiry, will the intervention affect their mathematical reasoning? Their attitudes toward math?

The Research Team

And so Math Trek was born, with the help of funding from a Javits grant for research on the gifted and talented, Office of Educational Research and Improvement, United States Department of Education. Math Trek was a complex effort by a team that included several senior investigators, but also a multitude of teachers and student assistants who contributed significantly to its outcomes.

The several senior investigators, most from the University of Washington (UW), came to the project with different backgrounds and agendas. Nancy Robinson, Ph.D., UW Professor of Psychiatry and Behavioral Sciences and Director of the Halbert Robinson Center for the Study of Capable Youth, brought a long interest in the early emergence of precocity in development and a knowledge of psychological testing. For example, with several colleagues, she had previously followed, to school age, early-talking toddlers (Crain-Thoreson & Dale, 1992; Robinson, Dale, & Landesman, 1990) and other preschoolers nominated by their parents as precocious in any of a variety of domains (Robinson & Robinson, 1992), using psychometric measures to document their progress. Virginia Berninger, Ph.D., Professor of Educational Psychology and head of the UW School Psychology Program, brought further expertise in psychometrics, individual differences, and research methodology. Robert Abbott, Ph.D., UW Professor of Educational Psychology, brought a strong background in conceptualizing and analyzing psychometric data. Yukari Okamoto, Ph.D., Assistant Professor of Psychology at the University of California at Santa Barbara, brought a background of neo-Piagetian theory and measurement of children's conceptual structures, as did Robbie Case, Ph.D., of the Ontario Institute for Studies in Education. These investigators constituted the part of the team who directed major efforts at measuring and analyzing the cognitive aspects of the children's development.

Other members of the team brought expertise in teaching and focused on the intervention aspects of the project, the Saturday Clubs. Swapna Mukhopadhyay, Ph.D., Assistant Professor of Curriculum and Instruction, is a specialist in teaching young children mathematics, and provided much of the inspiration for the approach we used and much of what will appear in succeeding pages. Barbara Waxman, Ph.D., was completing her dissertation on young children's reasoning about math. Barbara was head teacher in the Saturday Clubs and, as authorship of the volume attests, became its voice as well.

The Research Plan

In the spring of 1993, a publicity campaign was mounted throughout the Puget Sound region, the metropolitan area surrounding Seattle, to find young children, then of preschool and kindergarten age, who were thought by their parents and/or teachers to be "good at math." Letters to schools, meetings with Head Start teachers and those involved in Washington State's own Head Start equivalent (the Early Childhood Assistance Program), articles in local newspapers, and radio talk-show interviews were the means we used. Rough guidelines were mentioned such as, "Asks questions about numbers or time." "Makes up games using numbers, such as playing store with prices." For preschoolers: "Uses adding and subtracting up to 5 and understands that these are related; knows that a dime is more than a nickel; plays board games involving counting spaces." For kindergartners: "Makes small purchases; wants to learn to tell time; reads symbols such as plus and minus; reads speed-limit signs; may understand that multiplication is shorthand for adding."

To investigate parents' ability to identify these children, a sub-study was conducted by one of our students (Pletan, Robinson, Berninger, & Abbott, 1995). Pletan sent a questionnaire to the first 120 families who nominated kindergarten children. This questionnaire was developed on the basis of parent comments about their children's early interest in mathematical ideas. The questionnaire can be found in Appendix A.

The identification process resulted in the nomination of 798 children, with 778 families actually mailing in information and consent forms and permitting their children to be screened. We tested every one of these children with two or three brief standardized arithmetic measures. In the screening process, testers administered the Arithmetic subtest of the *Kaufman Assessment Battery for Children (K-ABC)* and the Arithmetic subtest of the *Wechsler Preschool and Primary Scale of Intelligence, Revised* (*WPPSI-R*). Children who had reached their sixth birthdays and who hadn't topped out on the *WPPSI-R*, were also given the Arithmetic subtest of the *Wechsler Intelligence Scale for Children, Third Edition* (*WISC-III*).

From the 778 children we screened, we first selected all the children who had attained a score at the 98th percentile or higher on any of the screening measures. We also included four boys with lower scores but compelling evidence of special interest in numbers. Because this group was too large for our resources, of the 348 children who met these inclusion criteria, we kept all the girls but randomly dropped 9 preschool and 29 kindergarten boys. The final sample of 310 children included 61 preschool girls, 78 preschool boys, 77 kindergarten girls, and 94 kindergarten boys. According to their intended kindergarten and first-grade placement, a total of 158 different elementary schools were involved! Half the children were then randomly assigned to either a comparison or an intervention group, with the provision that all the children who attended a single school were in the same group.

One arm of the study was directed at documenting and following for two years the cognitive development of all 310 children (Robinson, Abbott, Berninger, & Busse, 1996; Robinson, Abbott, Berninger, Busse, & Mukhopadhyay, under review). In the fall of 1993, and again in the spring of 1995, the children were individually administered a battery of measures tapping not only a wide variety of mathematical reasoning functions but also verbal ability, visual-spatial ability, and short-term working memory span; they were also asked questions about academic self-concept taken from the *Pictorial Scale of Self Perception* (Harter & Pike, 1984) and, in the final round, questions about what they found satisfying in math, some of these our own queries and some borrowed from Nicholls and his colleagues (Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990). Embedded in the battery were experimental measures of conceptual structure developed by neo-Piagetian investigators (Case, 1985; Crammond, 1992; Okamoto, 1992a, 1992b; Okamoto & Case, 1996); these measures as well as the Harter questions were also administered to the children half-way through the study, so that special analyses of their growth patterns could occur. Some families had moved out of state and a few had dropped out of the study before it was over. We actually re-tested 284 of the 310 children seen originally.

The other arm of the study is of greatest relevance to this book. For the children in the intervention group, Saturday Clubs were offered every other Saturday during the two succeeding school years, a total of 28 sessions in all. In groups of about 15, with the guidance of certified elementary teachers most of whom had been trained by Professor Mukhopadhyay, children met for half-day sessions, either morning or afternoon. In these sessions, the children were offered opportunities to become engaged in a wide variety of math activities. Most teachers were assisted by two, sometimes more, people, some of whom were graduate-student teachers in training and some of whom were undergraduate psychology students. Various sites were provided to be as convenient as possible for this geographically far-flung group of families. For the most part, the younger group (kindergarten and then first-grade as the study progressed) met in the morning and the older group (first-grade and then second-grade as time went on) met in the afternoon, but adjustments were made for the children's other activities (soccer games, ballet lessons, and chess tournaments were major events), car pools, and, in some cases, maturity levels. There will be much more about the Saturday Clubs in the chapters to come.

Research Findings

The results of our work have been described in three research articles (Pletan et al., 1995; Robinson et al., 1996; under review). Our first question had to do with identifying the children. We knew from previous studies that parents of very young children, even toddlers, could identify children who were advanced in talking, reading, and general intelligence, but what about math? The results of our initial search showed that, indeed, parents can identify math-talented children. On average, for the 778 children we screened, mean standard scores on the Arithmetic subtests of the *K-ABC* and the *WPPSI-R* placed them at the 86th to the 90th percentile. Had parents not been good at identifying the children, the mean standard scores should have been at the 50th percentile. The children we picked to follow as a group of course scored even higher. Their mean standard scores on the *K-ABC* arithmetic subtest placed them at approximately the 95th percentile and at about the 98th percentile on the *WPPSI-R* arithmetic subtest.

How accurate were the parents as informants? The substudy (Pletan et al., 1995) in which we sent a questionnaire to 120 parents of kindergarten children demonstrated that, indeed, parents were not only good at picking the children but accurate in describing them. Five factors characterized the parents' responses to the questionnaires: a general intellectual factor, a short- and long-term memory factor, a rote (rehearsed) memory factor, a spatial reasoning factor, and a specific relational knowledge factor. Scores on the first three factors correlated significantly with screening scores on both scales, but scores on the last two factors did not.

At what other cognitive tasks are young, math-talented children advanced? We were not surprised to find that the children were advanced on all the standardized measures we administered, not just the math subtests. Their scores on the measures of verbal ability, the Vocabulary, Comprehension, and Memory for Sentences subtests of the *Stanford-Binet, Fourth Edition* (*SB-IV*), hovered around the 90th percentile. The same was true of their scores on the two *SB-IV* visual-spatial subtests, Pattern Analysis and Matrices.

Within this pattern of advanced abilities, however, some abilities were more closely related to math than others. Specifically, the children's scores on the various visual-spatial measures (the two above, plus a visual-spatial working-memory span measure) were much more closely correlated with their math reasoning scores than were the scores on the verbal measures. This was a little less true for the older group than the younger group, but the pattern was the same: For young children who are good at math, visual-spatial reasoning abilities probably play a very important role in the way they think about math, a more important role than does reasoning in words or purely verbal knowledge.

Are there gender differences even at this early age? Alas, there are. More boys than girls were nominated for the study, and of the children nominated, the boys tended to score higher. Indeed, within our group we had a hard time finding as many girls in the younger group as we wanted despite extra appeals. We finally settled for an incomplete group (61 versus the 77 we had hoped for). In contrast, we found a superfluity of boys for the older group. And, within the battery of scales we administered, there were significant gender differences in favor of the boys on seven of nine math reasoning measures, including one memory measure. Contrary to expectations, there were not apparent gender differences on the visual-spatial reasoning measures, but there was a difference on the one visual-spatial memory measure, in favor of the boys. Neither the verbal reasoning nor the verbal memory measures were characterized by consistent gender differences.

When we looked at just the highest-scoring children, those in roughly the top 5% on each measure, the same gender patterns held—the boys were again overrepresented in this top group on the mathematical reasoning measures. The pictures for the verbal and visual-spatial measures were more mixed and not so gender-lopsided.

We didn't know what to make of this finding. We certainly had no definitive explanation of the gender differences. We doubted that parents were simply overlooking girls who were good at math, since in that case, those they identified should have been, on average, at least as advanced in math as the boys who were identified. We speculated that, whether or not there were any built-in gender differences in mathematical reasoning, the children's life experiences might well have influenced their number knowledge and number sense. Little boys, it's easy to observe, tend to gravitate in their play to countable and structured materials like blocks, cars, and railroad tracks; little girls tend to be more social in their play (Maccoby & Jacklin, 1974). We suspected that parents might tend to engage sons more frequently than daughters in talk about numbers. Is it possible that the gender differences we saw were wholly the product of children's differential experience? Or could some built-in differences have determined the children's play choices and colored the way in which the adults responded to them? In this way, could initially very small built-in differences have been magnified during the preschool era?

We waited anxiously to see how the children fared over the next two years, when they were tested again at the end of first and second grade. Indeed, as a total group, in all three domains we examined (verbal, visual-spatial, and mathematical), they not only remained as advanced over their agemates as they had been the first year, on some mathematical reasoning measures, they were even more decidedly ahead. This was true of the comparison group that had not attended Saturday Clubs as well as the group that had. Among the standardized subtests on which we could compare them with children of their age in the general population, at the last testing they were even more impressive in their abilities to extend series of numbers arranged according to principles they had to discern, to visualize mathematical problems spatially, to answer questions about the number system, and to do written calculations. In visual-spatial reasoning tasks such as copying block patterns or completing matrices, they had also made advances relative to their age peers. We concluded that not only were these children ahead of their peers when we first saw them, but also as a group, they held their own in every respect and made even more impressive gains over the next couple of years.

What about the gender differences? We found that these did not disappear. Indeed, in overall mathematical reasoning, the boys made greater gains than the girls over the two years, although the girls' progress was greater on one subtest involving word problems. Although we had expected that the girls, once exposed to classroom instruction, might compensate for possible earlier differences in their play experience, this was not the case. We seemed to have discovered another instance of what many call the Matthew Effect ("the rich get richer") (Walberg & Tsai, 1983).

With regard to the impact of our Saturday Club intervention, the overall effects were statistically significantly positive when we looked at the strides made by the intervention group on the math-related measures compared with those made by the comparison group. Recall that the children had been randomly assigned to the two groups, so there was justification in attributing the differential gains to the intervention. The intervention group had made the greatest gains in geometric reasoning, but some positive effects of the intervention were seen on almost all the math measures. The difference between the groups was not significant in terms of their growth on the verbal measures and only marginal on the visual-spatial measures, tending to confirm the notion that Saturday Clubs had specifically facilitated the mathematical reasoning of the participants. We were, of course, pleased with these findings. Indeed, it was these findings that gave us the courage to share with teachers the philosophy underlying our approach and the methods we used.

CHAPTER 3: Alternatives in Meeting the Needs of Math-Advanced Children—A Smorgasbord

In this book, we want to help teachers rethink the ways they can meet the needs of children who are capable of more advanced mathematical reasoning in the context of their classrooms, whether in homogeneous (what a misnomer!) classrooms for gifted children or heterogeneous, inclusive ones. Most of the book will introduce or re-introduce largely constructivist teaching methods and content that derive in part from those who have gone before (e.g., Duckworth, 1996; Kamii, 1984, 1989); these writers have all emphasized, as we do, teaching for meaning and "big ideas," as well as empowering learners.

This approach is also fully consistent with the *Curriculum and Evaluation Standards for School Mathematics* published in 1989 and subsequently elaborated by the National Council of Teachers of Mathematics, emphasizing active learning, problem-solving, reasoning and number sense, pattern-finding, engagement with manipulatives, questioning, and communicating ideas. Contemporary classrooms are characterized by real-world thinking rather than rote practice with computational operations and memorizing number facts (although these have their place). New methods of instruction encourage children's ability to grasp meaning in the number system, to see patterns and relationships, to figure out more than one strategy to solve meaningful problems, and to communicate their findings and discoveries to others. Children often work together in small groups, problem-solving with a variety of materials (manipulatives, formal and informal devices for measuring, calculators, computers); talking, writing, and drawing about their ideas; examining ways in which their ideas differ from those of their classmates; developing their own questions and problems; and otherwise participating in activities that are much more lively and engaging than math in the "olden days."

Such an instructional approach enables teachers to become intimate with what children are thinking, to become question-askers rather than answer-tellers, and to help children become engaged in problems that are meaningful to them. It is the kind of atmosphere in which children's thinking thrives. Most of this book is devoted to promoting just that kind of interaction.

First, however, in this chapter, we will outline some alternative means of adapting to the needs of children whose mathematical thinking is advanced, and so present a smorgasbord of educational options. *Nothing in this book should be taken to imply that there is only one right way to deal with the needs of these children (or any other children, for that matter). There is only one wrong way: To ignore the fact that children differ, to treat them as all alike, to be inflexible, to rob them of joy or confidence.*

Not only are there many ways to meet the needs of math-advanced children, but each child needs a mixture of possibilities and experiences—choices creating a full plate selected from the smorgasbord. A rich buffet consists of many dishes from which teachers and children can select several to match appetites, skills, deficiencies, curiosities, risk-taking proclivities, and so on. Nobody takes just one dish!

A Guiding Concept: The Principle of the Optimal Match

Every philosophy of education that takes into account individual differences among children works on the notion that, at any time, there is an optimal level of instruction that captures a child's readiness to learn: an optimal match. Only if children are engaged in learning at a level appropriate to their ability and skills, at a level they are almost ready for, is there likely to be a real change in their ways of thinking. Significant learning involves stretching one's mind.

If things go too slowly, boredom and turn-off are almost inevitable; if things go too rapidly or at too high a level, children (especially young children) are likely to become uncertain and avoidant. For math-talented children, the dangers are usually in going too slowly or in too shallow a fashion, with unneeded repetition of what the children already know and too little incentive to become truly engaged with new concepts, to figure out, to experiment, to see connections, to make sense of things. Children who are bored do very little learning. This is especially true of young children, who are almost all inveterate hedonists, ruled by what feels good and what doesn't.

Underchallenged math-talented children who are by nature well behaved may not let their teachers know how turned-off and miserable they are. In contrast, those who misbehave in response to similar feelings may be so irritating that their teachers never guess the underlying source of the trouble. (One very bright child of our acquaintance was expelled from first grade for kicking the teacher out of sheer frustration and four years later entered a university from which, a model of deportment, he graduated with a B.S. in mathematics at age 13.)

We do occasionally meet young, math-talented children who have been pushed too quickly through mastery of mathematical procedures, generally by their parents rather than their teachers. Typically, these children don't show signs of being in love with math. They may be able to carry out advanced operations, but without much conceptual understanding and with very little joy. One father told us that his son, then in kindergarten, was doing his "17 times" tables, whereas in fact the child was completely nonplussed by simple word problems using numeric relationships below 10, and couldn't produce number facts he had not memorized. Even worse, he scowled and mumbled when faced with math tasks and clearly wished he were somewhere else.

For the most part, however, parents are both good informants and sensitive to their children's interests and levels of understanding. What may seem at first to be a "pushy parent" is much more likely to be a parent running to keep up, striving to produce an optimal match. These parents can do an even better job if teachers validate their perceptions, respond to their children's needs, and enlist their collaboration.

There are many ways to produce an optimal match in the classroom for math talented children. The essential aim is to pace instructional experiences so that they fit the child's intellectual and personal maturity, thereby producing appropriate challenge, supporting growth, and enhancing the student's energy and motivation. Achieving an optimal match requires flexibility, ingenuity, compromise—and effort. The Principle of the Optimal Match, as you've guessed by now, is appropriate for all students, not just those with advanced capabilities.

Fundamental Versus Complementary Components

One useful way to conceptualize components of an optimal match setting for children is to distinguish between those aspects that are part and parcel of the basic instructional program (fundamental components) and those that embroider upon and extend it (complementary components). This book basically addresses the former, in essence, the regular school day. Some of our attention will be directed toward creating a climate and presenting activities in which children of all levels including the math-talented will be engaged with active learning about math and supported in spontaneously reaching their own optimal match levels. Even when we discuss supplementary activities in which children can engage in individually or in small groups, activities that extend the regular curriculum, for the most part we will assume that these activities occur during the regular school day or those after-school parts of it that we call "homework."

There are other possibilities for complementary activities that a school might provide, although in practice most of these tend to be limited to older children. There are, for example, inter-school, regional, and even national academic competitions in mathematics, but few of these involve children in the primary grades. A few younger children will be ready "early," however. The contest for the youngest children of which we are aware is Mathematical Olympiads for Elementary Schools (Dr. George Lencher, 125 Merle Ave., Oceanside, NY 11572). Group activities such as Odyssey of the Mind can begin with kindergarten on a local level; math is embedded in many of the tools the student groups may use to problem-solve in this competition. An interested and a talented parent might start an after-school math club that, for example, seeks out patterns and problems in the "real world." There are many such worthy possibilities capable of enriching and extending the experience of talented students, but such complementary activities cannot make up for inappropriate instruction during the regular school day. We owe children joy and challenge and the power of doing something new and difficult—every day.

Acceleration

When children are advanced in any area, their growth has been faster than expected for their calendar age. That's what the term means. To achieve an optimal match, therefore, the challenges presented them must likewise be advanced.

Mathematical knowledge and reasoning, particularly in the primary grades, have basic aspects that are linear and orderly in their sequences. Early-taught skills and strategies are needed in order to proceed to more advanced tasks. Furthermore, even if it weren't absolutely necessary, we do teach topics in relatively predictable order (e.g., we usually teach children fractions before decimals). Within a single class, there are children at many steps along these developmental sequences, with the math-advanced children generally (but not always) leading the way.

For these reasons, it makes good sense to assess each child's developmental status, keep track as the year progresses, and create an optimal match by proceeding along the regular sequence in what is offered, pacing advancement by the child's readiness. This process is called acceleration, but it does not imply that anyone is pushing a child (as one pushes the accelerator to make a car go faster). Even when we offer children what we think of as enrichment, deepening and broadening the range of their mathematical concepts and activities beyond the basics, it's important that these activities match their developmental levels. Advanced enrichment is probably a better term. (Never give more problems at the same level just because a child works quickly!)

For many math-talented children, out-of-level assessment measures are needed—tests that are designed for children somewhat older, sometimes considerably older, than most children at their own grade level. For math-advanced primary-grade children, a good place to start is two grades ahead. Some assessment can be quite informal—for example, chapter tests from textbooks borrowed from a teacher in a higher grade. If nationally standardized tests are administered regularly, these children can be given—in math, at least—measures designed for older children. More efficient is the use of a broad-scale standardized screening measure such as the *Woodcock-Johnson Tests of Achievement, Revised (WJ-R)*, which the school psychologist may be willing to administer. From this, one can get a ballpark grade-level estimate of the children's math skills and some information about specific skills in both calculation and problem-solving, creating a sense of their "frontiers of development." Some children are basically a chapter ahead; some are a grade ahead; a few are light-years ahead.

Such assessment measures won't tell teachers everything. They won't tell how deeply a child understands a concept. For example, a child who can calculate perfectly well using zeros may not have a clue as to what zeros stand for or how difficult life would be without them. Children who can easily pick out squares and trapezoids may not know their defining features. And a child who can solve a problem using a practiced algorithm may not have enough understanding to get to the same answer by a different route. An imaginative teacher, observing and asking questions, can develop a fine-tuned notion of a child's mastery of a concept, a very useful complement in the day-to-day classroom to the bird's-eye view afforded by standardized tests.

And test results certainly won't reveal the specific problems a child is already working on "in her head." Teachers have to listen carefully to children to find out. Sometimes a passing remark will be a clue, when least expected. A notebook and pencil in a pocket to jot down such comments can provide a reminder of a topic to return to or a clue about a child no one had ever suspected of harboring a mathematical inner life.

Smorgasbord Options Within the Classroom

No matter how mathematical talent is determined, teachers must still figure out how to meet the children's intellectual needs. There are, in fact, a great many ways to accomplish acceleration within fundamental instructional plans for young children. Here are some.

Compacting the Curriculum

Compacting achieves a productive balance for each child of time spent on the regular curriculum and activities that stretch and extend the curriculum beyond its usual boundaries. For the child who grasps concepts quickly and acquires facts easily, it is appropriate to reduce the number of problems or exposure time to a minimum in favor of other activities that extend concepts and skills to higher levels. A number of authors (see especially Reis, Burns, & Renzulli, 1992; Starko, 1986; Winebrenner, 1992) have presented effective ways of compacting the curriculum as well as making good use of the time thereby saved.

Even before a teacher introduces a concept, some (or many) of the children in the class may already understand quite a bit about the topic. The teacher's task is to assess what they do understand in order to know where to go from there. For example, before starting a new chapter, children might be permitted to take the end-of-chapter test. Children who already know the material with something like an 85% level of mastery can profitably skip it or concentrate on the parts of the chapter they haven't mastered. There's no real reason to limit this practice to the children who are thought of as math-advanced. Some of the others—indeed, the whole class—may do surprisingly well. Unfortunately, many current textbooks present a good deal of content that most children in fact have already mastered, and such pretesting can avert what would otherwise have been essentially wasted effort. On the other hand, for the child who is exceptionally advanced in mathematical reasoning, even going quickly through grade-level chapters may not be appropriate.

Working Ahead in the Curriculum

If pretesting demonstrates that a child already knows the material, the teacher can continue with chapter testing until a concept or skill pops up that needs work. This is the simplest step to take, but it can be one of the most troublesome if the child progresses (almost inevitably) beyond the textbook for his or her grade. There is a dilemma here: By providing an optimal level of instruction one year, a teacher may risk an even worse experience for the child in subsequent grades. It is essential, therefore, to plan ahead and negotiate with future teachers and school administrators so that the child is not expected to repeat the same work later. That would be the most devastating of all possibilities! Yet, it is untenable to argue that a child should be held back from learning what he or she yearns to master because a bureaucracy is too inflexible to adapt!

It is important, too, not to abandon children to their own resources as they work independently; keep track and keep in touch. Otherwise learning may lag or become superficial, and children will resent the lack of attention.

Mentoring

For primary-age children, sometimes a math-interested parent, a high school or college student, or even a student from one of the upper elementary grades can be invited to help the young child with math-related activities for one or more hours a week. Every teacher needs extra hands, and young children usually relish 1:1 contacts, especially with a Big Kid. Our only caution here is to beware of the older gifted children's being imposed upon for these purposes to the detriment of their own opportunities for learning. Don't be oversold by the studies of older mentors who have profited from mentorship as much or more as their mentees; often, the older children involved have been those with learning difficulties themselves, and the benefits to older gifted children may be much more limited.

If older children are used as mentors, or math-talented children are used as mentors for classmates, there are ways to make the experience worthwhile for them as well. Children can, for example, profit greatly by coaching in teaching techniques such as question asking, wait time, using alternative explanations, and so on. Feedback sessions will give them a chance to reflect on their skills as teachers, help them value and learn from the activity, and make them more appreciative and observant of their own teachers!

Diagnostic Testing Followed by Prescriptive Instruction (DT-PI)

A specific model of accomplishing acceleration for children advanced in mathematical thinking, a more formal version of the last two options described, is outlined for elementary school students by Lupkowski and Assouline (1992). These authors, translating for young students the approach developed by Dr. Julian C. Stanley in his Study of Mathematically Precocious Youth (Stanley, 1990), describe mentor-based programs working from formal pre- and post-test assessments. The DT-PI is a useful approach to acceleration with math-talented children in elementary grades and provides an interesting complement to the approach we will describe here.

Learning Contracts

Learning contracts are written agreements between teacher and child (or a group of children) establishing the parameters of an independent plan, including working conditions and acceptable behavior on the child(ren)'s part. Usually, this will include some parts of the basic curriculum, some advanced enrichment activities that extend the concepts being taught, and some free choice. Children can, in fact, often identify quite accurately what they need to work on, and they can certainly say what they'd like to work on.

Generally speaking, it's a good idea to keep contracts related to the saved-time domain, in this case, math. It may not be "fair," for example, to punish math-talented children by having them spend extra time working on subjects with which they have trouble. There are, however, creative ways to use mathematics to make connections with tasks the children favor less. One might, for example, have children practice spelling with interesting math words, read biographical material about mathematicians or scientists or about the history of mathematical discoveries, or inventory the manipulatives or science equipment in the classroom (to practice handwriting). Certainly, if the child is eager to work on a major non-mathematics related project during contract time, there is no reason not to encourage this.

Math-advanced children, just like other children, need the kinds of skills, strategies, and knowledge that take repetition, although they may need less practice than the others. Many bright children are more interested in concepts than skills. Once they understand something they've had explained, they are impatient to move on—but they may not "own" the concept or skill quite yet. One might ask the child to explain or write rules or draw pictures so that another child could understand. The teacher will surely embed practice with skills and strategies in the assigned problem-solving activities. But math-advanced children shouldn't be permitted to talk their way out of learning number facts and procedures to the point that they become automatic, that is, capable of being retrieved from memory without having to be figured out each time. This automaticity is a great asset as one solves new problems, for it reduces the effortfulness of the process, and enables the child to keep focused on the ideas rather than the details. A child who has number facts readily available will be more likely to detect relationships among numbers; for example, the numbers 63 and 81 will immediately be seen as members of the "9 family" by the knowledgeable child. Once these number facts are solidly in place, however, nothing much is to be gained by further practice.

Activities to Extend the Math Curriculum Without Driving the Teacher Crazy

There are a great many sources of ideas for math-related advanced enrichment activities aside from those teachers invent for themselves. One great place to start is the Addenda series of publications from the National Council of Teachers of Mathematics, some of which are listed together with other resources in the Annotated Bibliography at the end of this book. In addition, teacher and children can devise a whole array of problem-solving activities using materials already at hand or some the child can bring from home (e.g., sports pages from the newspaper, utility bills, road maps, small foreign coins).

Figure 1 presents a few ideas of assignments that could be used to extend concepts being taught in class. The extensions included here range widely in difficulty level—just as do the abilities of "math-talented" children in the primary grades. They are presented as a way of getting teachers started thinking of ways to embroider upon an existing curriculum in order to provide advanced enrichment possibilities for their students. In addition to such teacher-generated ideas, however, it is well to remember that the children may have excellent ideas of their own. Most math-talented children have an accurate idea of what they do and do not know. One might, for example, ask children who would clearly be underchallenged by a worksheet of simple addition facts to make up a harder set of problems for themselves, or if there are two children who are working at about the same level, to make up problems for each other. The same children may be good sources of ideas for projects to accomplish, though they will need help in reviewing the projects for feasibility and modifying them accordingly.

CLASS IS LEARNING	ADVANCED ENRICHMENT ACTIVITY
Combinations to make 10	* Using the 4 operations, find how many ways one can make 10 from combinations using the number 2. * Write equations for these. * Extend the above to make other numbers (5, 12, 41, 64, 0, -2). * Using Cuisinaire materials with the orange rod = 1 rather than 10, what is each of the others worth? On graph paper, graph these combinations. * Combine dice to = 10. For each number, 2-12, find how many combinations one can make with 2 dice to equal that number. What is the probability of getting each combination? Each number?
Adding single digits	* Plan a trip to the state capitol using highway maps. Freeway versus state routes? Other trips? * Use missing addends: If traveled this far, how much is left? * Rate: How many hours by freeway? State routes? Stop for lunch?
Subtracting single digits	* Reframe subtraction as adding negative numbers. Think of as many examples as you can of subtraction and/or negative numbers in real life and make up story problems. * What is the difference in age between the oldest and youngest child in the classroom?
Estimating	* Make estimating jars for class, filling with manipulatives, or any available objects. Have students bring objects from home. Figure out strategies to increase accuracy (e.g., weight, volume, length). * Ask: What makes estimating easier or harder?
Rounding numbers for place value	* Adapt a board game (e.g., Parcheesi) by requiring that a problem card be answered before each turn. * Individualize pack for each child or group. Try 999 or 9,972 to nearest 10 or 100; 4 x 3 x 2 to nearest 10; -8 to nearest 10. Use fractions, decimals, rounding down.

Figure 1. Ideas for expanding math curriculum after compacting.

(figure continues)

CLASS IS LEARNING	ADVANCED ENRICHMENT ACTIVITY
Exploring 1/2, 1/4	* What is a "quarter?" How much is a quarter of: an hour, a mile, a kilometer, a quart, a cup, a liter, a moon, a year, a dollar, a roll of quarters? * Cut an apple in quarters. Does each weigh exactly the same? * Explore thirds, fifths, and sixths. * Find real-life contexts for fractions (e.g., music, cooking, making change). * Explore multiplying fractions using pattern blocks (e.g., 1/2 of 1/4). * Make patterns with halves and quarters of shapes. * Use geoboards and tangrams to explore fractional parts of shapes (e.g., 1/2 of a rectangle = a triangle).
Dividing by single digits	* Bring in family's utility bills. Average per month? Season? Per person? * Collect small foreign coins, if available. Look for value in newspaper and monetary system in World Almanac. Coin's worth in cents? Dollar's worth in coins? * Play "store" with foreign prices.
Mosaic patterns	* Tessellating or quilting patterns. * Analyze tiles in floors or elsewhere in school. * Use computer software to generate tessellations.
Computation worksheets	* Have children figure out why they made the mistakes they did (bugs) or look over class papers (names removed) to find most frequent bugs and report to class (use frequencies and histograms to present data). * Have children write ways to check answers (complementary procedures). * Have children write story problems from these computation problems or make up more complex problems and write those in story form. * For multi-digit addition problems provide the answer and leave blanks in the two addends.

Figure 1. Ideas for expanding math curriculum after compacting (continued).

Smorgasbord Options Between Classes

The next several possibilities involve matching children with classrooms to achieve an optimal match.

All-School Math

Using the Joplin Plan (sometimes known as "All-School Math" or "cross-grade grouping"), a school can achieve an optimal instructional match for all students. Everyone in the school does math at the same time; classes are arranged in order of advancement within the curriculum. The developmental steps between groups are smaller than full-grade steps, permitting students to move ahead (or behind, if need be) without making great leaps likely to create gaps.

Such an approach requires all teachers to collaborate in creating a developmental continuum for instruction, with children placed almost irrespective of their home grade. The highest group in the school has no limits in instructional level except those appropriate for the children. Because the students in a given group are all at about the same developmental level in math, the teacher and children can target a topic more effectively than if the children are ability-grouped within the classroom, the teacher spending time with only one group at a time (Gutierrez & Slavin, 1992).

This approach does, however, require the cooperation of teachers and administrators in agreeing to when and for how long math will be taught. The problem of the older child whose math achievement is at a low level has to be handled tactfully. Finally, the plan doesn't lend itself well to integrated instruction across disciplines, although that can occur at other times during the day. Yet, its advantages are many and it deserves serious consideration.

Cluster Grouping

Arrange classroom assignments so that the most highly capable children are placed in one classroom (or possibly two classrooms) at each grade level. In a school with three or four classrooms per grade, typically, the most capable students are distributed among all the classrooms, complicating both their lives and their teachers'. Cluster grouping facilitates the brighter children working together cooperatively, sparking each others' ideas, and giving their teacher a chance to work with them as a small group on more advanced material. Cluster grouping is informal and can change from year to year. It is well suited to children who are more advanced in some domains than others. In cooperative learning groups in which more advanced children work mainly with those who are less advanced, the advantage is almost wholly for the latter, but cluster grouping of math-talented children (see Kennedy, 1995) assures that peers who can indeed provide advanced ideas and provocative questions for one another have a chance to do so. Yet, because the children constitute only one cluster within a heterogeneous classroom, they accrue the social advantages of being with a diverse group of classmates.

Ability Grouping Within the Classroom for Core Instruction, Especially for High Ability Students

Faced with a primary classroom full of children whose developmental levels easily range as much as five grade levels, most teachers already do some ability grouping for reading and math instruction. Because it is so familiar, we needn't discuss this practice here except a reminder that, even within the highest third of students, the range of achievement can still encompass several grade levels. Compacting, contracts, acceleration, and enrichment in a variety of combinations will still be needed.

Multi-Age Classrooms

Multi-age classroom grouping is a general term for several different approaches to teaching. "Split grade" classrooms generally maintain the distinct curriculum of each grade and can have real advantages for gifted children who are thereby exposed to older children and more advanced curriculum, but only if they are in the lower of the grades involved. To spend two years in a 2-3 split classroom has little long-term advantage, but to spend successive years in 2-3, 3-4, and 4-5 classroom organization may have. Eventually, one would expect the child to move up a grade, and this structure provides a gentle path to that outcome. Otherwise, the child ends up essentially repeating the final grade in the sequence.

Other multi-grade approaches present, at least for much of the day, a common curriculum for all the children, although not all are expected to respond at the same level. This kind of approach can potentially provide unlimited "top" for children, but in fact the mixing of ages usually succeeds in so increasing the heterogeneity of the class makeup that providing for individual children taxes teachers severely. Many multi-age classrooms use interest centers that can give children appropriate options, but the most advanced children tend to "use up" such options very quickly. Even among the talented, the less confident children may not choose the more challenging options unless asked to do so.

Trading Students: Subject-Matter Acceleration

If a child in a second-grade class would be better instructed in one or more subjects at the third-grade, fourth-grade, or fifth-grade level, perhaps a colleague could be talked into that possibility.

One shouldn't limit one's imagination! We became acquainted with one eight-year-old several summers ago when a professor in the University of Washington Department of Mathematics called us for advice after discovering the boy in his calculus class. The third-grade teacher of this Vietnamese former boat-child had seen to it that he could complete precalculus courses at a nearby community college the previous year. When he applied for summer school, submitting his transcript, it was assumed that the birthdate must be a typographical error! A combination of subsequent full-time placement in a challenging program for gifted children and tutoring by a high-school calculus teacher to encourage playfulness with practical applications of higher math, permitted this child to complete second-year college instruction in math as well as some science classes by the time he was 11. How many of us would think this was possible, or healthy? It was both, as this friendly, exciting youngster proved as he developed into a strong, warm, gentle, happy, and high-achieving young man.

At the same time, especially in the primary grades, one needs to take into account whether a young child can really keep up with older students in ways other than the central abilities for which a teacher is trying to find a match. One needs to consider whether, for example, the child has the fine motor skills to keep up with the written computation expected and whether the expected level of reading is appropriate. Some allowances may need to be made or extra assistance given. Especially in math, it may be easier to find a good fit in this way than it would, for example, in a writing class, but it's asking for trouble just to plunk a student down in a higher class and expect everything to work out automatically.

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Early Entry to Kindergarten or First Grade

Most educators are leery of enabling children to cross age barriers, as though calendar age was the ultimate piece of knowledge one could have about a child. Age is, of course, important, but is it all we need to know to place a child in school? Consider that our laws generally dictate that, unless we go through lengthy procedures to override the system, we must admit to kindergarten a child who may have been born three months prematurely on August 31 (due perhaps in late November) and must exclude a highly capable child, born full-term on September 1, who understands multiplication and is reading at the fourth grade level. (States vary in their cut-off dates.) Is this a reasonable position? The research evidence (see Robinson & Weimer, 1991) overwhelmingly demonstrates the wisdom of welcoming into kindergarten or first grade children who can't jump the birthday barrier, but are otherwise mature, intellectually advanced, at least average or better in fine and gross motor skills, and on their way to reading and computation. Careful assessment is necessary and there are many factors to consider—but early entrance is clearly a viable possibility for helping to achieve, for a while at least, an optimal match for children who are eminently ready for school.

Skipping a Grade

While generally less limited by law than in making decisions about early school admission, most educators resist double promotion, or grade-skipping, no matter what children's academic achievement levels are and, indeed, no matter what their social skills or the ages of the friends they spontaneously seek out (Jones & Southern, 1991). Most bright children do seek older friends and share with them academic interests as well as play interests, hobbies, and ideas (Robinson & Noble, 1991). Advancement by one grade when a child is young may take up a good bit of the slack between grade placement and developmental level for a child who is moderately ahead. Kindergarten, first, and second grade are good candidates for skipping when children's academic skills are quite advanced, since in most schools, third grade sees the introduction of cursive writing, more complex thinking about math, and transition to more abstract aspects of reading, with most basic skills firmly established. Later on, eighth grade is often a good choice, since its curriculum is generally not particularly distinctive and it constitutes the transition year before students transfer to high school.

The research about grade acceleration is overwhelmingly positive with respect to capable children whose advancement is in several domains, not just math (Kulik, 1992; Kulik & Kulik, 1984; Rogers, 1992; Rogers & Kimpston, 1992). Children's academic achievement on average profits to the extent of acceleration—those who are accelerated by a grade achieve a whole grade higher than do equally bright children who are not accelerated. And what about their mental health? The evidence here is again very consistent. There are no overall differences in the self-concept and mental health of children so accelerated, compared with their non-accelerated peers. One might argue, indeed, that since the accelerated children's social comparison groups are composed of children older than they are, they might be expected to see themselves less favorably, a "littler fish in a bigger pond" rather than "bigger fish in a littler pond" (Marsh, 1987). If their self-concepts are comparable to those of non-accelerated bright age-peers even under these conditions, they can be seen as doing very well.

Grade skipping won't create an optimal match forever. Children who are a grade ahead as they enter school will be several grades advanced later on. For example, a child whose development is roughly 25% more rapid than that of her agemates may, at age 6, be a grade or so advanced but, at age 12, about three grades ahead. More immediately, primary-grade children who are distinctly advanced in their mathematical capabilities will still need attention and advanced enrichment opportunities, since their skills may be considerably more than a single grade ahead.

Pull-Out Programs and Resource Rooms

In some school systems, children who are advanced in one or more domains are "pulled out" of their own classrooms for special group instruction, typically from an hour to a day a week. When the curriculum is sufficiently deepened, differentiated from, and advanced beyond that of the regular classroom to accommodate the needs of the children, such placement can be effective and healthy (Delcourt, Loyd, Cornell, & Goldberg, 1994). Such programs can incorporate mentoring and long-term project development that are difficult to provide in a regular classroom. In fact, however, such pull-out programs seldom provide core instruction in mathematics and it is still the job of the regular class teacher to make the kinds of adaptations we've talked about.

Furthermore, some pull-out programs, because they must adapt to children from so many different settings, tend to resort to "fun and games," extra field trips, and non-academic activities to keep children's interest high. In doing so, they not only court jealousy from other parents and students for whom the activities would be equally appropriate, but do not provide sufficient academic pay-off for bright children to make up for the disruption and expense they cause the system. In an atmosphere like the current one, with tight school budgets and political issues about "elitism," pull-out programs (which are always expensive) are highly vulnerable.

Special Classrooms

This is not the place for a discussion of the provision of self-contained classrooms for highly capable children. We point out, however, that in large districts, such programs may well be the easiest way to establish settings in which children's advanced development can be enhanced. In such classrooms, the "normal" pace of instruction is more rapid; the basic curriculum can be covered quickly by all the children; and the kinds of extensions engaged in by only a few children in a regular classroom will be useful for everyone. Children in special schools and separate classes show substantially higher levels of achievement than both their gifted peers not in programs and their gifted peers attending within-class programs, though they may be somewhat more reliant on teacher guidance (Delcourt et al., 1994).

While the range of abilities in self-contained classrooms will still be very high—because the most advanced children may be several grades ahead in one or more domains—the open-ended strategies described in the remainder of this book, strategies that can simultaneously engage children at several levels of competence, are likely to be less demanding of teacher energy when the range of ability goes from above-average to sky-high rather than below-average to sky-high! Self-contained classrooms, especially those with class sizes equivalent to those of other classes, are, of course, much less costly than pull-out programs, since they utilize the basic education budget. Furthermore, they are less likely to be political targets because it is clear that the children are working at least as hard as, often harder than, their agemates, on learning tasks that would overtax other children.

Teacher Consultants/Enrichment Specialists

Some school districts, both rural and urban, successfully employ specialists to assist regular classroom teachers in planning and executing activities appropriate for gifted children, right in their regular classrooms. Supporting the classroom teacher by bringing books and materials and, above all, new ideas, and sometimes by brief instruction of the children themselves, these specialists can make a significant difference in the ability of schools to meet a child's needs. Two of the nice features of such an approach are its flexibility and the opportunities it furnishes for teacher and specialist to brainstorm and plan. Children need not be formally selected or labeled as gifted, and children with uneven development can be readily accommodated. Often, there will be one or more unlabeled children who are attracted to the special activity and who thereby reveal themselves as advanced in ways that had not been previously apparent. Staff specialists are add-ons to the school budget and therefore not always available within a district, although such persons often may be found in state or regional service centers if teachers look for them.

Open-Ended Strategies in the Classroom

Thus far, we have touched very little on the real life of the classroom, the interaction of teachers with children, and children with children. It is in the climate, the community, the shared delight in the discoveries of the mind, that a setting is created in which learning can occur. And it is in what Kennedy (1995) has called a "gifted-friendly classroom" that children will want to learn. In a gifted-friendly classroom, gifted children feel valued and comfortable, protected from teasing and from the expectation that they are in all ways "perfect." Teachers in gifted-friendly classrooms are not compelled to limit their vocabulary, jokes, and ideas to those that all children will grasp, but occasionally float ideas and tasks that only some children will appreciate. Teachers in such classrooms also encourage learning for its own sake, reward children for struggling with complex ideas, and hear children out when their ideas are incomplete or initially seem off-target. Gifted-friendly classrooms are friendly places for all children.

What we will describe in Chapters 4 through 7 is, like the options discussed above, part of a smorgasbord of options for math instruction. It is, however, also an underlying agenda, for the strategies we will describe can help to create child-friendly classrooms in which problem-posing and problem-solving are open ended and suitable for the simultaneous engagement of children at many levels of mastery. In creating a climate of warmth, acceptance, safety, and respect, a community of learners who track and take pride in their own mastery, but do not measure their worth by comparing themselves with others, teachers can set the stage for such an open-ended approach capable of challenging advanced learners. Such classrooms are fun for both teachers and children and constitute good places to grow.

Conclusion

Achieving an optimal match for math-advanced children can and should take many forms. This smorgasbord of strategies merely scratches the surface. As mathematics education is being reformed (e.g., National Council of Teachers of Mathematics, 1989), its greater flexibility, its multiple goals and strategies, and its real-world orientation are all beautifully suited to a learning community in which math-talented children can thrive. Teachers who watch and listen to the individual children soon give up their preconceptions of what children in a specific grade "are like," preconceptions often created by adherence to linear, underchallenging, and unimaginative textbook-driven curricula.

Math-talented children are particularly fun to listen to. In every classroom there lurk children who are in love with math, or who can be, and whose capabilities just don't fit the mold. They are very much worth the trouble it takes to teach them well.

CHAPTER 4: The Math Trek Curriculum—Philosophy and Practice

Rachel and Tanya had just worked for about an hour on a project they had devised, figuring out how many chips of each color there were in the Chip-Trading Game, and how many points that added up to for each color. (See Appendix B for a description of this game.) Together, they ran over to their teacher to share their findings. Their teacher marveled with them over their discoveries and the strategies they had invented to keep track of their data and then suggested other problems to pursue. She stayed with them as they worked on these problems, some of which involved visualizing squared and cubed numbers with Multilinks (cubes that link together in multiple ways). The level of energy and excitement was high. As Rachel played with solutions she turned to her teacher and said, "This is great! I never knew math could be *so many things*."

To some extent, this example symbolizes the goals of Math Trek: To nurture advanced mathematical talent by helping children to become autonomous in their learning, to work well together, and to share their mathematical discoveries with each other. How did these children come to be able to pose a problem, immerse themselves in that problem, devise strategies and flexibly use materials to represent solutions, and find such joy in the process?

Certainly the children themselves played a role; they came to us curious and interested in math, more knowledgeable about the mathematical system than other children their age and, most of them, open to our ideas and activities. The teachers, for their part, were open to and curious about the children and their intellectual interests and diverse ways of approaching math. There were also some very conscious ideas and a body of research that informed the philosophy and guided the practice of the Saturday Clubs. As mentioned previously, Dr. Swapna Mukhopadhyay, a mathematics educator from the University of Washington, was in charge of the training of Math Trek teachers. She conveyed great respect for children's thinking and sense-making abilities and offered many creative activities and questions to pose to the children. She also inspired the teachers to create innovative problems and to ask children probing questions that would, in turn, inspire their own thinking.

The authors' philosophy was informed by their own research into children's mathematical thinking in the area of negative numbers, stories for equations (Mukhopadhyay, 1995; Mukhopadhyay, Resnick, & Schauble, 1990) and place value (Waxman, 1996), as well as by their reading of Duckworth (1996), Kamii (1989, 1993), Cobb and Wheatley (1988), Schifter and Fosnot (1993), Gardner (1983, 1989), and Rogoff (1990), among many others. What follows is a quick summary of key ideas.

Beliefs About Learning

Duckworth (1996) writes about four beliefs that are essential for children to develop about themselves if they are to be effective learners: a) The "way things are" beliefs, for example, that both addition and multiplication are commutative (i.e., 3 + 4 is the same as 4 + 3), but division and subtraction are not; b) "It's fun" beliefs, as in, it is enjoyable to shrink and stretch the number line and to see what happens when you try to think of all the numbers between 0 and 1 or all the numbers between -50 and +50; c) "I can" beliefs, such as "I can continue the Fibonacci sequence into the hundred thousands place"; d) "People can help" beliefs as in, "If I'm stuck on this problem I can ask another child, the teacher or an aide."

Multiple Abilities or "Intelligences"

Another important set of ideas comes from Howard Gardner's (1983) concept of multiple intelligences, that is, the idea that intelligence is multifaceted and that each child has strengths and weaknesses in different domains. Given the nature of this study, we knew that the children were all strong in the logical-mathematical domain. We quickly found out on our own what the testing also revealed, that most of these children were also advanced in the visual-spatial and verbal domains. Gardner identified other domains of intelligence including the musical, the kinesthetic, the interpersonal and the intrapersonal, domains in which this group was more variable.

By involving multiple domains in learning, children are able to strengthen each type of intelligence as well as to understand more deeply the particular concept or skill being studied. For these reasons, we encouraged the children to talk and write and draw and enact their understandings of particular concepts and by doing so, to make use of their verbal, visual-spatial, and intrapersonal (reflective) intelligences. In one class, children put on a play about probability. In several classes, part of recess was devoted to a kinesthetic enactment of numerals as well as lots of estimating activities ("How long do you think it will take you to run across the room?" or "Count how many hops it takes to get you to the other side."). Children were also asked to keep records of their work using drawing, writing, or a combination of both. By representing their knowledge in different media, children become engaged in a different sort of problem-solving, one that encourages a deeper understanding of the subject (see Chapter 7). And, as an added bonus, asking children to represent their knowledge also encourages the formation of the beliefs about learning discussed above.

Play and Playfulness

Another important idea informing our work with the children was the importance of both play and playfulness (Mukhopadhyay & Waxman, 1995). By *play*, we mean several things, from plenty of time just to "muck around" and explore the qualities and relationships of materials and activities, to systematically exploiting variations and permutations, to practicing a newly acquired skill. Lainie, for instance, came in each week with her latest trick to memorize the multiplication table, which she would then practice for many minutes at a time. Joni, as described below, spent the first half hour of every session constructing a pattern with whatever materials were available. That each pattern was a take-off on her last one cued us that Joni was methodically exploring the structure of patterns. And Peter delighted in varying the mathematical challenges posed in board games. For instance, when playing Monopoly, he doubled the value of the properties and based all his calculations on the doubled property value. (No doubt he'll be prepared to live in a city with expensive real estate!)

By playfulness, we mean an attitude toward mathematics that entertains all possibilities, including humorous ones. Indeed, there was much laughter during the Saturday Clubs as children made connections, struggled with concepts, and shared discoveries. For instance, during a discussion of right angles, one child piped up, "So, what are left angles?" During one of Swapna's visits to a class, the children decided they wanted snacks of carrot sticks for a change. Swapna posed the question, "If we were to have five sticks each, how many carrot sticks would I need to bring in?" The children readily figured out the answer, generously including Swapna as well as the teaching staff. Swapna then posed a variation on the problem: "Well, how many should I bring in if we were each to get seven carrot sticks?" By posing this question, Swapna not only made them work harder (seven is far harder a number to compute with than five), she showed them that one can vary a problem in a playful yet challenging way. And she followed up by bringing to the next session the number of carrot sticks they calculated! Playfulness, then, was not just the province of the children; the teachers also manifested a playful spirit about math, thereby setting the tone for the classroom.

Problem-Posing

Closely related to the idea of play is the idea of problem-posing, that is, inventing a problem or challenge as did Peter, Rachel, and Tanya (see above). These inventions tell us much about children's intellectual agendas, their often hidden mental life. All children need some unstructured time to get to know materials and "play" or "mess around," and they need to return to play at different stages with the same materials. Very often, when observing children's messing around, you will see that they are posing problems for themselves, and that the kinds of problems they pose change in complexity over time. Thus, children's problem-posing gives us a great assessment tool. By examining the sorts of problems children invent for themselves, we can readily see their current mathematical agendas, that is, the nature of the concepts and skills they are attempting to master. By observing children's problem-posing we can more readily create the optimal match discussed in the previous chapter.

Problem-posing is not limited to materials or games. Children also pose problems by extending problems given to them, so long as those problems are open-ended and challenging enough to suggest other avenues to pursue. Indeed, problem-solving and problem-posing are intertwined processes and arise from the same rich contexts. One of our tasks, then, as teachers, was to figure out how to construct those rich contexts and which strategies were most conducive to eliciting children's problem-posing. We found that collaboration with peers, allowing dead-time, acknowledging frustration, and supporting experimentation through questions and hints fostered the children's problem-posing.

Creating an atmosphere in which children's problem-posing is respected and allowed to flourish also leads to *cognitive empowerment* and to *finding one's own intellectual voice*, key ingredients in nurturing mathematical talent (Mukhopadhyay & Waxman, 1995). By cognitive empowerment, we mean the capacity to take one's own ideas and others' ideas seriously and to be able to sustain inquiry into challenging, problematic, and even confusing issues. By finding one's own intellectual voice, we mean developing a line of inquiry, an approach to problem-solving and a style of expression (e.g., a favored medium such as drawing or use of patterns) that becomes almost a signature of that child. The remaining chapters of this book are studded with examples of cognitive empowerment and children finding their own intellectual voice, especially in the character profiles in Chapter 8.

Sense-Making, Model Building, and Understanding

Another premise that guided our practice is an obvious one, but one that should never be taken for granted: that these children (and all children) are engaged in making sense of mathematics (Kamii, 1993). They came to us, young as they were, with many ideas and intuitions about mathematics already formed—ideas about number, numeration, operations, and shapes. Our task, then, was to figure out what sense they were already making of mathematical topics, and to engage them in further sense-making activities. We also wanted these children to become comfortable communicating their sense-making to themselves and to each other.

One feature of children's sense-making involves the building of mental models that represent non-obvious phenomena they encounter such as electricity or barometric pressure. These models are often incomplete or inchoate and require a rich context, a community of learners, and good questions in order for children to explicitly formulate their models and to test them through explorations and experimentations. This process of articulating, testing and sharing enables children to become better sense-makers. As teachers, our goals were to provide the conditions for this process and to make sense of the children's sense-making. In order to accomplish the first goal, we provided materials, fostered a community of learners, developed good questions, and listened (and watched) intently for how the children reasoned, struggled, and questioned. Listening and watching also helped to accomplish the second goal, along with our journal writing and meetings where we discussed what we had observed.

We were also well aware that the road to understanding pivotal math concepts can be a long and sometimes frustrating one, even for math-talented children. While rote understandings of concepts and skills prove to be fairly easy for these children, a deep understanding of a concept occurs over a span of time and is often forged in stages. For this reason, we revisited concepts and big ideas in different ways at different times. Exploring the same idea in different contexts helps children to extend and generalize their understandings. This approach allowed time for what the psychologist David Elkind (1976) calls "horizontal elaboration," that is, thoroughly exploring and extending the meanings of a concept before moving on to the next level in the hierarchy. At Math Trek, we wanted the children to make the biggest ripples possible when casting their intellectual stones. In Chapter 6, we describe our method of revisiting and elaborating on concepts.

Inventing Procedures, Developing Number Sense

Much of the recent research in the development of children's mathematical thinking supports the idea that children can and should invent their own methods for solving computational problems (Duckworth, 1996; Kamii, 1993; Schifter & Fosnot, 1993). By inventing their own procedures, children are engaged in problem solving even when doing computations and they are forced to think about the meaning of the numbers instead of relying on rote procedures that allow them to disregard the quantities the numerals symbolize. Children's invented procedures reveal a great deal about their number sense, such as their ability to decompose and recompose numbers (for example, 37 + 44 is the same as 30 + 40 + 7 + 4 which is 70 plus 11 which is 81), thereby providing a good assessment device. In fact, inventing procedures helps children develop number sense for they have to figure out sensible ways to get answers. If children attempt to adhere to the rules for algorithms without understanding the conceptual basis for those rules and without thinking about what numbers and operations mean, then their computational mistakes may result in answers that demonstrate a decided lack of number sense. Therefore, at Math Trek, children were allowed the freedom to invent their own procedures. They were also asked to share their ingenious strategies with each other, which gave them the opportunity to analyze verbally why their strategies worked. Through this approach to problem-solving, the participants were not only building number sense, they were building a community of learners, too.

Defining the Teacher's Role

The teacher's role needs to complement this view of children's learning as inventive, sense-making, model-building, and based on what children already know. In this view, teachers are facilitators, guides, designers of challenging problems, and probing questioners. The teacher's role also involves *nurturing* and *sustaining* the children's problem-solving and posing activity. By nurturing, we mean allowing frustration and struggle to occur, and helping children to unpack and work through that frustration. In this way, the emotional component of the frustration is rendered a cognitive issue (e.g., "No I can't do this!" becomes translated into "Oh, I can't do this *because* . . . but I can try this instead."). By sustaining, we mean adding complexity to the problem situation so that children push the boundaries of their mathematical understanding. Teaching in this manner requires constantly being open to the ways children are thinking, and to various mathematical possibilities that could be pursued. In order to paint a fuller picture of what this sort of teaching looks like, we have filled the curriculum chapter with examples not only of what was studied, but also the kinds of questions the teachers posed and their responses to the children's mathematical thinking.

Teacher as Learner, Too

The Math Trek teachers saw themselves as learners of mathematics, too. Our own sense of what it's like to immerse ourselves in a mathematical problem, and our taking time to think things through both mathematically and pedagogically, helped us nurture and enjoy children's mathematical thinking. (Sometimes we ourselves encountered and worked through temporary confusion—as for example, when we first tackled chip-trading in reverse.)

The Role of Good Questions

One of a teacher's most important tools is the ability to ask good questions, questions that can elicit a child's current understanding or gently push a child to consider other possibilities. By virtue of asking the right question at the right time, a teacher can both figure out how a child is currently making sense of a problem and engender a cognitive conflict for a child to resolve. Sometimes, the most intriguing questions would arise from playful interactions with the children or during a whole group discussion in which everyone, including the teacher, would be fully involved in a mathematical perplexity. Because so many of the questions asked depended on the mathematical activity, the Big Idea being pursued, and the individual children involved, it is impossible to make a list of good mathematical questions. However, in Chapter 6, we give you the flavor of these questions as they are embedded in the descriptions of the curriculum we used and our interactions with children around big mathematical ideas.

A Word About Manipulatives

While manipulatives are almost ubiquitous in primary math classrooms, controversy exists as to the role, function, and use of math materials. At Math Trek, we were clear that manipulatives are tools for problem-solving and a means to represent and embody mathematical thinking. Therefore, we never enforced a particular way of using the manipulatives; children were free to use or not use the materials available, and to invent their own strategies for making use of them. Observing how the participants used the materials when left to their own devices provided important clues about their mathematical thinking, their insights as well as their misconceptions. Sometimes children, particularly math-talented children, are heard to say they dislike manipulatives and find that they hinder or slow down their mathematical thinking. However, when manipulatives are used flexibly and creatively to solve problems and to embody and communicate meanings, resistance to their use melts away and even very bright children become enamored with their possibilities.

Importance of the Social Context in Learning

While it is true that many talented young children are intensely creative and inventive in their mathematical thinking, it is also true that these children benefit greatly from working with peers and adults (Greeno, 1991; Lave & Wegner, 1991; Rogoff, 1990; Vygotsky, 1978). Peers provide a stimulating source of theories, ideas, strategies, and conflict. Adults, including teachers, aides, and parents, can play various roles, such as guide or master thinker. When children are working in a group, however small that group, interaction with peers or adults always involves thinkers at both similar and different levels. This verbal and nonverbal exchange allows children to hear ideas and strategies that they have not thought of previously. Some of the ideas may even be too advanced for direct teaching or immediate mastery. The encounter helps the child to reach somewhat beyond his or her current level of thinking, into what Vygotsky (1978) has called the "zone of proximal development." The more advanced peers (or adults) provide the necessary scaffolding to help the child reach to the next level. Thus, children come to know much about math on the social and interactional plane; in time, they will internalize the knowledge and make it their own. By the articulation they do in the group, the more skilled peers help themselves to internalize mathematical processes, for they are speaking as much to themselves as they are to others. In the next chapter, we describe the ways in which we consciously fashioned the social context or climate of the Saturday Clubs.

Conceptualizing Mathematics Broadly

Teaching math requires not only thinking about how children learn, it also requires thinking about the nature of mathematics. Mathematics is a rich, well-structured, and organized domain that possesses interrelated parts. Concepts do not exist in isolation, but are interconnected and embedded, thereby creating a coherent system (Scholnick, 1988). We hoped that the children would come to conceptualize the world mathematically, to see all about them numbers, patterns and relationships, and to make sense of what, when they were younger, were bits-and-pieces of mathematical knowledge.

What to Teach? The Power of Big Ideas

It is precisely because math is a rich and interconnected domain that we felt the Math Trek curriculum should focus on the central ideas that permeate many aspects of mathematics. We knew that in their own schools the Math Trek participants were getting plenty of exposure to the typical content of early mathematics curricula. At Math Trek, we had the opportunity to try out some big concepts that underlie typical curricula but are seldom brought to the surface, as well as to explore concepts that are often considered too advanced for young children. Thus, we structured much of the curriculum around big ideas such as *equivalence*, *reversibility*, and the *visualization of numbers*. The participants' growing appreciation of the richness and interconnectedness of the math domain attested to the power of these big ideas.

For instance, we explored the issues of numeration in different cultures. We also explored the ways in which number can be used to represent shape (e.g., the golden rectangle) and the ways in which shape can be used to represent number (e.g., the Vedic Square). Several themes were pursued periodically throughout the two years in different ways and from different angles. For instance, an important aspect of mathematics is reversibility, a process which takes many forms and which results from the logical structure of relationships that comprise mathematics (e.g., subtraction is the inverse of addition, division is the inverse of multiplication, equations lead to graphs and vice versa). One way to challenge children and encourage flexibility and alternative ways of examining things is to ask them to go in reverse (e.g., in the game of chip-trading, to start with the end-point and attempt to get back to the beginning point—see Chapter 6), or to provide children with the answer and have them pose the question as in, "If 14 is the answer, what is the question?" Sometimes we gave children an equation and asked them to create the story that would fit that equation. In Chapter 5 we describe in more detail many of the big ideas that we pursued in the Saturday Clubs, as well as the ways the children responded to and extended them.

Teachers' Beliefs About Mathematics

As teachers, we needed to develop our own appreciation of the richness and interconnectedness of mathematics, and to view math in as broad a manner as possible: as a way of perceiving the world, as something one finds everywhere, and as a way to describe our world. In this way, we came to see math as a humanistic endeavor as well as a scientific one. Concretely, this meant relating math to other disciplines such as science, art, and literature, and helping the Math Trek participants come to see math in these broad and rich ways, too. To that end, we spent a fair amount of time talking about what constituted mathematics and encouraging children to share their conceptions with us.

We came back to that question many times over the two years, approaching it in many different ways. For example, as discussed later on, at the end of each session we asked children to categorize their favorite activities by recording those activities in one of three books: number, shape, or logic. We also did an alphabet exercise in which we asked the children to identify a mathematical term for each letter of the alphabet, or to limit the list to particular concepts such as magnitude, shape, or number. Such activities generated a lot of excitement. At appropriate times, we asked the children how an activity we were doing related to math.

We also asked the children, "What is a mathematician?" in an attempt to have them identify with the thought processes and strategies involved in thinking mathematically. In discussing that question, children came to realize two things: First, that they were indeed mathematicians and, second, that being a mathematician has to do with thinking, strategizing, and encountering new ways to view the world, not just speedy right answers. In this way, we came to create a metacurriculum, the goal of which was to have children think about mathematical thinking in rich and flexible ways. In this way, we also helped the children to generate higher-order knowledge:

> . . . what makes higher-order knowledge higher order is its aboutness. Higher-order knowledge is about how ordinary subject-matter knowledge is organized and about how we think and learn. (Perkins, 1992, p. 101)

Aesthetics, Passion, and Transcendence

As part of our effort to define mathematics broadly, we also wanted to include elements not often associated with mathematics—aesthetics, passion, and transcendence. Heady words for such young children, but still part and parcel of a full and rich experience of mathematics! Take Joni and her pattern-creating. Every Saturday Club, Joni, with the assistance of her good friend, Lauren, would immediately set to work making a huge and elaborate pattern with the colored inch cubes. Joni was rather shy and did not speak much at all the first year of Math Trek. It seemed that allowing Joni to choose her very own intellectual agenda was enormously satisfying to her. Watching her work, one became caught up in how absorbed and intrigued she was. Her teacher asked her once if she thought about what she was going to do before she came in. She said, "No, it just comes to me when I sit down with the blocks." However, each time, she tried something new, some variation on a pattern, or a reversal of a pattern she had created before. Her strong aesthetic pull permeated her problem-solving style. One day, following a story reading (see Chapter 6), Joni and the other children tried to figure out how many more apple trees would be included next year when Farmer Jane made her square orchard one size larger than the 36-tree square she had planted this year. Joni used a color pattern that made the solution obvious to all the children. Joni's absorption in and enjoyment of her work, and her ability to emerge from her pattern-making to partake fully in the class all seemed to reflect what Foshay (1991) has entitled a *curriculum of transcendence*.

Creating the space for Joni to engage with the materials in her own way and in her own time had a remarkable effect on this inordinately shy and reserved little girl. By the middle of the second year, she was speaking up more often and gaining confidence in her own voice. Her confidence was much deserved and it was a delight to watch the other, more vocal and assertive children come to respect her. We learned later, in talking with her regular classroom teacher that, on the Monday following Saturday Club, Joni regularly shared with the class what she had learned and the activities she had performed at Saturday Club.

How would a child like Joni define mathematics? Brilliant though she is, it would probably be hard for her to articulate precisely her sense of mathematics. We will attempt to do it for her. Her conceptualization would include first and foremost the idea that there are meaningful patterns in the world and that people can create as well as respond to them. She would emphasize that mathematics is something one could find in stories, in art projects, in science experiments, in building projects, in shapes, and in numbers. For Joni, mathematics was also all about connections, the idea that there are connections among problems, among patterns occurring in nature, shapes, numbers, and even algorithms. Her notion of pattern and sequence helped her to explain to the other children how one can use the regrouping algorithm in subtraction with objects available in the classroom such as Base-10 blocks. She would also explain that mathematics is visual and aesthetic and something to be shared with others.

Open-Ended Curriculum

Given our conceptualization of mathematics and our view of how children make sense of mathematics, open-ended activities characterized by multiple entry points and no ceiling on sophistication were the natural choice for nurturing mathematical talent for the children in the Saturday Clubs. This strategy was designed to take advantage of the fact that we had a luxurious amount of time to spend on math (two and a half hours) and to help meet the diverse needs of a group of children with a two year chronological age range and various talents and proclivities. An open-ended curriculum is also useful with a wide range of children, but particularly for those at the advanced end of the spectrum. The process of open-ended curriculum also affords the space and time to play.

Structure of the Saturday Clubs

While these children loved math, the truth is that these sessions were voluntary "extras" for which they had to get up early on Saturday mornings or give up their Saturday afternoons. Often, their parents worked during the week and even at home time was precious. Therefore, it was incumbent upon the teachers to make the sessions as enjoyable and as enticing as possible. To that end, we worked hard not only on finding interesting activities and promoting positive group dynamics, but on structuring the time so that children were thoroughly engaged.

When the children arrived, they found an assortment of materials laid out on each table. At each table there were also new job cards that detailed particular problems or activities. The children were expected to choose two job cards each session, to carry out the task or solve the problem on the card, and to record one or both of their activities in their journals. Often, children would choose to work in pairs or small groups and to collaborate on their problem-solving. If time remained, they were encouraged to pursue any quiet mathematical activity with materials or books not currently in use. After this initial activity time, which lasted 45 to perhaps 60 minutes, the group came together for a meeting during which children were invited to share any discoveries or perceptions they had while completing their job cards. Over time, the children came to enjoy this process and willingly shared their stories, creations, and problem-solving strategies. Some of the children liked to make presentations together.

After the sharing time, the teacher generally posed a problem or activity for the whole group to work on together, for about 30 minutes. After this, the children were ready for recess which often included, for those who wanted it, a mathematical component such as timing runs, keeping track of rope jumping or ball bouncing, "Mathematical Mother May I" or solving more physical problems (e.g., everyone links hands, twists around and then figures out how to untwist).

After recess it was snack time, yet another opportunity to embed mathematical activities in a real-life context. For instance, the children were asked to estimate the number of cookies in the box, each person's share of crackers, etc. Once the children were settled with their snacks, it was time for stories with a mathematical bent, of which there are no shortages in children's literature (e.g., the *Anno* series by Mitsumato Anno, stories by Marilyn Burns and many others that are described in Appendix C). During the second year of the program we used the *Open Court Real Math* series (for these first- and second-graders, using third- through fifth-grade level stories). *Real Math* provides a wonderful collection of math stories that all use the same cast of characters but widely vary the mathematical situations that the characters (and students) encounter. The stories provided many opportunities for the children to verbalize their reasoning and to problem-solve jointly. Sometimes, a few children chose to act the stories out for their classmates, or to demonstrate their reasoning by using the blackboard, manipulatives, or whatever props were at hand.

After snack, there was time for one more activity. At this point in the session it was very helpful to explore math through an art project or a science activity. The ending section was also a time for children to write or draw in their journals, a natural closure and a way to share their activities with the parent who came to pick them up.

In the remaining sections of this book we attempt to make this philosophy come alive by telling stories of our Math Trek Saturday Clubs. In the next chapter, we discuss how the teachers set the climate. We then give readers more of the flavor of these sessions by describing several open-ended activities that we used and how the participants responded to these activities. These activities can easily be adapted for use in a regular classroom, for each has the virtue of being suitable and interesting for children at a variety of levels. They may even help bring to the surface mathematical talent that was never suspected. We then describe ways to integrate math with other domains in the curriculum. Finally, we portray some of the different flavors of mathematical talent that we observed at Math Trek.

CHAPTER 5: The Culture of the Classroom

Picture a group of children sitting around a table eating a snack. The story to which they are listening includes many word problems that they're solving as we go along; each child contributes and has something to say. Sometimes a lively, good-natured debate emerges, the children articulating well-reasoned challenges to each other's point of view. At the end of the story the teacher asks them what all those problems had in common. The children squirm a bit. Finally, Jenny asks, "Why do you ask us questions that you know the answer to?" The other kids chime in sympathetically. The teacher thinks for a minute and says, "Well, I know what I think the answer is, but I don't know what you think or why you think what you think. The only way I can find out is by asking you." This answer seems to make sense to the children. After a short pause they start to answer the teacher's question; through this discussion the children come to a consensus that the stories all involve logic. One child insists that a character is illogical and should think things out better. Another child agrees that the character should be more logical. Yet another is thoughtful for a minute and says, "But it takes more than logic to solve these problems; it takes imagination. You see, you have to be able to picture it in your brain; that helps you to see the answer."

The above anecdote describes a group of seven-year-olds who have been meeting together 24 times over the course of two years. Our first meetings together were not quite so cozy, sharing, and thoughtful. Two children would never have spoken up at all; another child would have ventured an answer only if she was sure that she was right. How did they come to be able to share, take risks, and feel so comfortable? How did they learn that it was okay not only to question themselves and each other, but their teacher, too?

This question goes to the heart of a central issue in mathematics education and perhaps all of education: How to set a climate in which all children feel valued and respected, able to take risks, share their thinking, pose their own problems, and extend problems posed for them. Teachers have a great deal of leeway and control in terms of how they want to create the climate in their classrooms. They can find a way to create a climate that empowers all the children in their classroom to take risks and to be active, not passive learners. All kinds of decisions, even curriculum and instruction decisions, flow from the type of climate or classroom culture that teachers decide to set.

How to Set a Climate That Empowers

Of course, it takes time, and lots of it, to build trust and a sense of ease. In addition, there are helpful strategies teachers can use (see Appendix D). One important strategy is to ensure that everyone's voice is valued by giving every child in the class a chance to be heard. In some classes, it is mainly the boys who raise their hands assertively and demand to be called on. One technique is to say to the class, "I'm interested in what everyone in this class is thinking. So I'm going to wait until everyone has thought through this problem." In reality, some children's minds work faster than others; however, quickness does not guarantee brilliance. Some children, given enough wait time, manage to create wonderful problem-solving strategies and to come up with insightful perceptions. Waiting for everyone to generate an answer lets children know that something besides speed is valued.

Wait Time as Empowerment

If some children seem bored by the wait time, it's fine to suggest another problem they could be thinking about. Alternatively, you could ask children who seem to process more quickly to find more than one way to solve the problem that was posed. This issue also comes up when children play games such as chip-trading (see Chapter 6). Some children quickly develop strategies that allow them to figure out what chips to ask for and what trades to make. Other children, however, take their time or want to see all their chips before they ask for a trade. The quick children often tell the slower children what they should ask for and which trades to make. Permitting such behavior is demeaning to the child who takes a longer time but who is perfectly capable of ascertaining what trades to make, and is a license for rudeness for the quicker children. Again, we don't want children to equate speed with brilliance.

In addition to providing alternative questions and activities for quickly processing children to answer, it's also important to have a continuing conversation with children concerning good ways they can make productive use of their time without bothering others when finished with the question or assignment at hand. Bringing children into this discussion helps them take responsibility for their learning and their behavior. It also acknowledges their facility with solving problems and the potential boredom that accompanies such speediness. For gifted children in regular classrooms, this is an issue of enormous significance: How do they handle the usual pace of a classroom designed for children learning less rapidly than they do? All too often, boredom becomes a way of life when both they and their teachers could avert it.

One enterprising girl figured out a way to handle her boredom during a group discussion at one of the Saturday Clubs. While most of the children were fully engaged in this discussion, this girl was not. A bag of dice lay on the table where Laura was sitting. She carefully turned each die so that the "six" was on top. Then she lined up the dice and counted how many dice there were. Then she posed the following question to herself: "If there are 24 dice, and they are all on six, how much is that all together (i.e., 24 x 6)?" The discussion ended shortly after Lauren began to find her answer. A number of the other children became curious about her problem and elected to help her. There ensued a busy ten minutes of problem-solving for about six children.

This discussion can also include children's emerging theories of intelligence (see Dweck & Bempechat, 1983). Some children believe that being smart is simply a matter of ability (which Dweck calls an "entity" theory of intelligence), while other children believe that being smart is a matter of working hard (an "incremental" theory of intelligence). Clearly, a belief in working hard will get children further than a belief in ability, which can lead to merely resting on one's laurels. This is a particularly important message for math-talented children for whom many mathematical skills are gained so easily. Eventually, all children will bump into something they cannot immediately understand. What will they do? Give up? Or plunge in, determined to master what is initially mystifying? Open-ended activities also play a role in enlarging children's ideas about intelligence by providing the leeway to get to challenging mathematical problems without a lot of dead time. Providing time to share mathematical discoveries alerts children to multiple ways to solve problems and to varieties of mathematical creativity. In this way, children can come to see that being smart involves more than speed and getting good grades.

Thus, it is not only in overall level of instruction that children need the optimal match, it is also in the *pace of instruction*. Giving one child time to think—sometimes quite a bright child who is tussling with a difficult problem or a new way of approaching an old understanding—does not mean that everyone has to be left to his or her own devices. Because math is so interrelated and interconnected, children should have no problem extending or posing their own problems. If finished with seat work early, children can write in their math journals about what the easiest problem was, or the hardest, what would make an easy problem hard or what would make a hard problem easy. Children could also be asked to find a more elegant or simpler solution to their problem, or a simpler way of describing their solution. Children could also be asked to relate their current math activity to a real-life context, to make up a story problem for the equation they just worked on, or to give a justification for the procedure used.

Mess-Around Time Is Learning Time

Another important aspect of setting the climate involves giving children plenty of opportunity to play and mess around with materials and manipulatives before proceeding to introduce new goals. Davidson, Galton, and Fair (1975), the inventor of the Chip-Trading Game (as well as many other math activities), and Burns (see 1987, 1988), the prolific math educator, both agree that children need to have plenty of unstructured time as they are introduced to new material.

This opportunity to "mess around" gives children time to pose their own problems, make their own discoveries, and set their own agendas. It can also be a wonderful, informal opportunity to assess and tune in to one's students. For instance, a heterogeneous class of first graders is given pattern blocks for the first time that school year. A number of children attempt to build three-dimensional objects; one child insists on sorting and stacking the same kind of blocks; two girls make animals; several children explore how different shapes can be combined to create hexagons; and several other children build complicated patterns that involve symmetry. The fact that children pursued such different activities informs the teacher about the status of each child's development with regard to perception of shapes, congruency, patterns, and symmetry.

Extending Problems and Ideas

The children's self-created activities also suggest other avenues to pursue with them. For instance, for the children who made pattern block animals, a natural next step would be to invite them to write the instructions (or dictate them) for how to create those animals. Writing the instructions for the pattern block animals invites the creators to analyze their creations in terms of the kind and number of shapes used, thus extending their play into a more demanding task. After this is done, a child could give the instructions to a friend to see if the friend could make the animal from the instructions.

Leading Questions

For the children who are engaged in finding different ways to make hexagons, the teacher could pose one of the following: "Find all the ways to make different hexagons." "Which shapes can't you use to make hexagons?" "Can you make a big hexagon?" "Find all the ways to make trapezoids, parallelograms, etc." "Make all those shapes bigger." The children who made complex patterns involving symmetry might be handed a little mirror and asked to find the lines of symmetry and to predict where, on their pattern, they will find symmetry. In this way, children at all levels and abilities are engaged in worthwhile academic pursuits, teachers are able to track the children, and the curriculum can follow the children rather than vice versa. Most importantly, teachers have the opportunity to discover talent that may not emerge when math consists only of computations or assigned tasks. And children feel a wonderful sense of freedom to explore and make their own discoveries.

Another good example of providing an open-ended activity for primary school children is simply to bring in a big bag of real coins. The fact that they are real coins enhances their value for children and imbues the proceedings with more importance. Each child receives a bag that includes several of each denomination. The children are then left to their own devices to explore the contents. What would a group of first graders do with such a bag? Some children will begin by sorting the coins and counting their stacks. Other children will want to know how much money there is all together and attempt to add on by ones, fives, tens, and twenty-fives. Other children will pose addition problems for themselves using their total amounts and that of their neighbors. Other children will examine the coins, find the dates, and determine how old the coins are, an activity that involves some complicated subtraction. The observing teacher can make note of what the children are doing and the strategies they use. She can then pose coin riddles for the whole group, such as: "I have four coins in my hand and they total 46 cents. What kind of coins do I have?" The children can also make up their own coin riddles. These child-created riddles provide good assessment material.

Individual Differences

By this time, it is clear that even talented children differ markedly in their tempo, style, and approach. Respecting these individual differences, and even taking advantage of them also helps in setting a climate that empowers children. In Chapter 8 we discuss several children who vary in tempo, style, and approach to learning math.

To Praise or Not to Praise

One of the best ways to help children feel good about themselves is to take their thinking seriously. For many adults, it is almost a habit to praise children whenever they show their work. The result is that children habituate to this praise and dismiss it as meaningless or assume that everything that they do is wonderful. Praise can also end or cut off a mathematical discussion instead of extending it. If, instead of automatically praising, we respond thoughtfully to what a child says, the child will infer that he or she is doing some good mathematical thinking. In this way, children will also respect the adults with whom they work and take their own work more seriously. In addition, children might also develop autonomy in the realm of learning. In the following story we describe how one of the teachers helped to foster autonomy for two of the Math Trek participants.

Developing Autonomy

Cathy was a remarkably strong-willed child who went to a very traditional school. Number facts and computations were stressed above all else. So when Cathy came to Saturday Club she brought with her the skills and stresses from her school week. Her best friend Michelle came to the same session, and their behavior was remarkably predictable. No matter what activities were carefully arranged on the tables, they would march right over to the blackboard to play "Teacher." This game involved one of them putting a computation on the board and instructing the other child to solve it. After the problem was solved, the "teacher" would mark it right or wrong. One day, the two became involved in a big dispute over one of these board problems. The problem was this:

$$\begin{array}{r} 20 \\ \underline{\times\ 20} \end{array}$$

Cathy said that the answer was 40. We overheard Michelle say that didn't look right to her. Cathy insisted, saying that her father had told her. The two of them demanded the correct answer from the teacher. First, we asked Michelle what she thought the answer was. She said she thought it was 200, because she thought it had to be a lot more than 40, which looked way too small. Although Cathy insisted that we tell them the answer right away, we suggested that they determine it for themselves. We asked them which of the many materials around the room might help them figure it out. Cathy immediately ran to the base-10 blocks with Michelle in hot pursuit.

The two girls were a little perplexed at first. They weren't quite sure how to set up the problem. How do you show 20 times 20 with base-10 blocks? Their teacher sat with them on the floor and asked them what 20 times 20 really means. Michelle thought for a minute and said, "Twenty groups of 20, so there has to be 20 of these things (the tens sticks) in each group." Cathy furrowed her brow and said, "that would be way more than 20; that would be 20 tens. I think you need twenty of these little things (the ones cubes)." Michelle became excited and said, "Oh, I see, but you don't need the ones, you can just use two of these (the tens)." Now it was Cathy's turn to be confused; she counted up each little cube on the tens block. As soon as she got to 20, she quickly began making piles of two tens each.

It took the two girls some counting and recounting, but they finally put together twenty piles of 2 tens each. When they finished, they sat back and simply surveyed all the blocks for a minute. Then began the task of counting up all those piles. Cathy's idea was to count by 20s. She began, "twenty, forty, sixty . . ." and stopped. "This is hard!" Michelle jumped around a bit and said, "We could just count by tens, see, ten, twenty, thirty, forty, ..." As she counted, she touched each tens block. But Cathy disagreed. "That will take too long!" Michelle deferred to Cathy and they again began to count by twenties. At this point a couple of other children chimed in to help with the count. Their teacher, satisfied that the children were finding a way to the answer, went off to check on another group of children.

About five minutes later, Cathy and Michelle ran over breathlessly to announce to their teacher that the answer was 400. She asked which of their answers had been closest to the actual answer. Cathy said, "In a way, mine was, 'cause if you look at it written, 40 is closest to 400." "How is that?" Cathy said, "Forty just has one less 0." Michelle then said, "But my answer is closest if you think about the number; two hundred is closer to four hundred than forty is." And with that they were off to the blackboard to give each other more problems. Only now, instead of running to us to verify answers, they ran to the base-10 blocks to verify, and for the next several months this was their new way of playing "Teacher."

This episode engendered several changes for these girls. First, it helped them to see that they could become their own authorities and did not have to rely solely on external sources such as the teacher. Second, these girls realized that they could solve problems collaboratively. Third, they developed an intense interest in base-10 blocks.

Talent Is Not a Guarantee of Immediate and Complete Comprehension

One lesson we quickly learned from the talented participants in Math Trek is that these children were not immune to misconceptions in mathematics, that at times these children were mystified by the written representation of arithmetic, and that even for these children there were some concepts that needed plenty of time and lots of play to come to a deep understanding. The following anecdote, involving the same two girls described above, illustrates how important space and time can be in rich mathematical learning. This anecdote also attempts to portray the constant need for teachers to make decisions as to how to respond to children.

Cathy and Michelle's interest in the base-10 blocks soon expressed itself in a game they devised. One session, they made a structure with the thousands cube and the hundreds flats. Then they took the tens blocks and pushed them through a platform in their structure. They seemed very busy and engaged, so we chose simply to observe instead of changing the game into a math problem (e.g., "How much does your structure cost all together if hundreds are worth a $100 each and thousands are worth a $1,000 each?"). They chatted away as they played and decided to call their game "Lumber Logging." We looked at each other and simultaneously said, "No, it should be 'Number Logging!'"

It was a curious game, and one to which they returned for a portion of each remaining session of Math Trek the first year. This game did not, on the surface, seem to have a great deal to do with math, or at least with big ideas in math or with pushing the envelope of the girls' knowledge and expertise in math. However, they were intent on developing and playing this game, and not to be dissuaded with other pursuits or challenging questions. Once they had played the game to their satisfaction, they were quite relaxed and ready to do the many other activities and problems that comprised the sessions. These girls were devoted to their Saturday Clubs and bemoaned the interval of two weeks between sessions. Cathy was particularly enamored of one project we assigned to the children: finding all the ways to express 14 for our end-of-the-year cake. The number 14 was chosen to commemorate the fact that we had met 14 times that year. Cathy fluently came up with addition, subtraction, and multiplication problems, so we asked if she could also find a division problem that yielded the answer of 14. She informed me that she had not been taught to divide yet. This discussion took place as snack was being distributed, so we quickly turned this activity into a division problem. There were four children at the table and the snack was goldfish. We asked the group, "Well, if there are 32 goldfish here, how many goldfish will each of you get?" Cathy commanded the bowl of goldfish and began to divide them up among the four children. She quickly figured out that the answer was eight. When asked how she had figured it out, she replied, "Simple. It was like dealing cards; you just count up how many you have at the end." We informed her that what she had just done was division! She was very impressed with herself and turned her attention to figuring out a division problem where 14 was the answer. By the end of snack, and with some collaboration from the other children at the table, Cathy came up with 28 divided by two.

We ended the first year of Saturday Clubs in April of 1994 and resumed the Saturday Clubs in September of that year. Cathy and Michelle came bounding in to the room that first Saturday delighted to see each other and their teacher. However, most of all, they were delighted to see the base-10 blocks ready and waiting for them. They turned to each other and said, "I know, let's do Number Logging!" This time, we observed that they had added a few rules so we suggested that they could write down the rules so that other children could play their game, too. This suggestion seemed to please them, for they immediately jumped up and got some graph paper and began to discuss the rules. We turned our collective attention to other matters that session, but the two of them left saying that they would finish their instructions during their sleepover that night.

The two girls assured us the next time that they had worked on their book of rules, but had forgotten to bring it in. They continued to play Number Logging all through this second year, too, sometimes bringing in other children into their play. Of course, they also pursued many other topics, too, as discussed in other sections of this book.

When the last session of the second year rolled around, the two girls came into the classroom looking a little sad and a little nostalgic. But they were also excited for they had remembered to bring their Number Logging Rule Book. They proudly announced that the book was a present for us and pointed out that they had dedicated the book to us. We read the book with them and complimented them on how clearly they had described the rules they had made up.

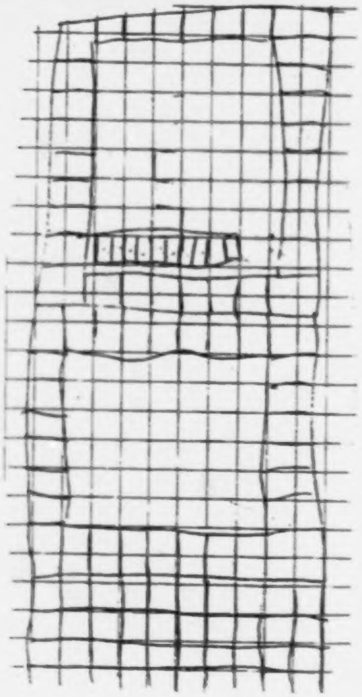
As if they were already reliving the good old times, they then rushed over to the rug where the base-10 blocks were and began their play. They called us over after they had played a satisfying game to show us how they were arranging the tens. Michelle had arranged her tens to show the word "TENS!" (she used a cube for the point of the exclamation mark). And Cathy demonstrated all the complements of ten (10-0, 9-1, 8-2, . . ., 0-10) by starting with ten in one group and none in the other group and moving one at a time to the second group.

As Cathy demonstrated this progression lovingly, she verbalized that you end up with the same amount you started with, just on the other side, and that along the way, the same groupings are created (e.g., 1 and 9, 9 and 1). Cathy's demonstration let us know at least one of the meanings this oft-repeated game had for her: It was a working out of all the additive components of ten as well as a way of deepening her knowledge of tens as a unit and as a multiunit (i.e., a unit composed of other units). Of course, she probably "knew" some of those meanings before she even began Math Trek. How obvious it seems that ten is composed of ten ones, yet this is not necessarily obvious to children. Math Trek allowed Cathy to play with and consolidate those meanings until they were coherent in her mind.

When their play was finished, the two girls carefully arranged the base-10 blocks in their box and again called us over to see how neatly they had done their work. Michelle proudly said, "That's our best put-away job ever."

Thus, setting the climate also involves constant readjustments of the thermostat and this is accomplished through our actions, responses, and reflections as teachers. In the next chapter, we tell more stories of what we learned from the Saturday Clubs, including the major themes that girded our curriculum.

Number Logging Rule Book
Hold your logs up and put them through the holes in the tower.



CHAPTER 6: Curriculum—Big Ideas and Many Extensions

This chapter describes much of what we learned from the Math Trek Saturday Clubs that could be useful in other settings. The stories that follow describe in detail many of the Big Ideas we pursued in Math Trek and how the teachers and children engaged with these ideas and with each other. What follows, then, are Math Trek stories. This chapter is not designed to be a set of curriculum recipes; it's designed to impart a strong sense of what we learned from conducting the Saturday Clubs and from the participants, and to reveal the nature and purpose of open-ended activities and their usefulness in working with math-talented children. It should also be noted that many of the mathematical themes were pursued over many sessions; they were not one-shot deals or encapsulated activities. We hope that this chapter will spark teachers' own pedagogical and mathematical creativity and impart a sense of how to find and pursue the big ideas in math with their own classes. Of course, there may be specific activities in this chapter that teachers will want to use in their own classrooms. To make them as accessible as possible, we have provided sequential descriptions of these activities, as well as questions and extensions in Appendix B. It might be very helpful to try out the activity and to play with the mathematical ideas involved before reading the chapter. As always, teachers' own continual learning engenders empathy and understanding of the learning process; it may also generate thought-provoking questions to ask of students.

What Is a Numeral?

To begin an open-ended activity, project or topic, start with a question that invites children to bring their own considerable knowledge to bear on the issue and to become reflective about the workings of some aspect of their world or the world of mathematics. For instance, a good topic for children in the primary grades is the issue of numerals, or written numbers. Numerals, along with letters, quickly become part of the taken-for-granted symbolic world in which the child lives. Indeed, for many children, letters and numerals are considered pretty much the same—special marks on the page. Because written representation is so important in mathematics, it can be very useful to explore the nature of these written symbols with children. In doing so, their theories and misconceptions will become evident and easier to work with.

The first year of Math Trek we explored numerals in many ways. One day, the children were asked: "What is a numeral?" Here are some of their responses: "A letter." "A number that you write down." "Something to add or subtract." "The numbers from 1 to 9." "Zero is a numeral, too." Also explored were their ideas about what numerals are used for, why they look the way they do, etc. The children were quite animated as they discussed all the ways numerals are used in our daily lives. They also played with the ways in which numerals symbolize the number that they represent by creating numeral characters. For instance, Mr. Three had three heads, three arms, three legs, etc. Showgirl 2 made the most of her curvy lines. And 3-bird's design included lots of sideways 3's (a universal stylization of birds).

The following year, a slightly different question was asked when the topic was revisited: "Why do we have numerals?" In this discussion, children were vociferous in their notions that without them it would be too hard to keep track of things. The teacher suggested just using tally marks. One child was disdainful of that idea: "That would get real annoying, real fast." How come? "Because then you'd have to count all those marks just like the things the marks are for." The idea of using slashes for bundles of fives was quickly suggested and found to be far superior to mere tallies. Such discussion helps all children to see the utility of grouping—an important idea in coming to understand our place-value system (Jones & Thornton, 1993).

What Is a Number System?

We also asked why some kind of numeral system might have been invented by human beings. This question directs the children's attention to the fact that mathematical symbols are human inventions as well as to the idea that mathematics has a history. Seeing mathematics as invention and as history subtly cues children that they, too, are capable of inventing. After this discussion, other numeral systems (Egyptian, Babylonian, Roman) were introduced. For example, Egyptian numerals were presented with some geographical and historical context, and in a way that allowed children to ferret out the structure of the system.

When presented with the Egyptian numerals, the children very quickly wanted to know if there was a zero and were aghast when informed that, "No, no one had invented zero yet." We then talked about the uses of zero. The children were asked to interpret various numbers symbolized by Egyptian numerals and to compare how many symbols are used in the Egyptian system versus the Hindu-Arabic system. They wrote their ages and their parents' ages, but were overwhelmed when they attempted to write their telephone numbers. The children were then asked to choose a partner and to give each other addition and subtraction problems using the numerals.

One of the perplexing aspects of the Egyptian numeral system for young children is that while it encodes powers of ten, it does not have place value or make use of the positional property. This contrast with our Hindu-Arabic system forced the children to reflect on the usefulness of place value as they played with the idea that one can arrange Egyptian numerals any way one likes. One teacher wrote 43 in Egyptian numerals like this:

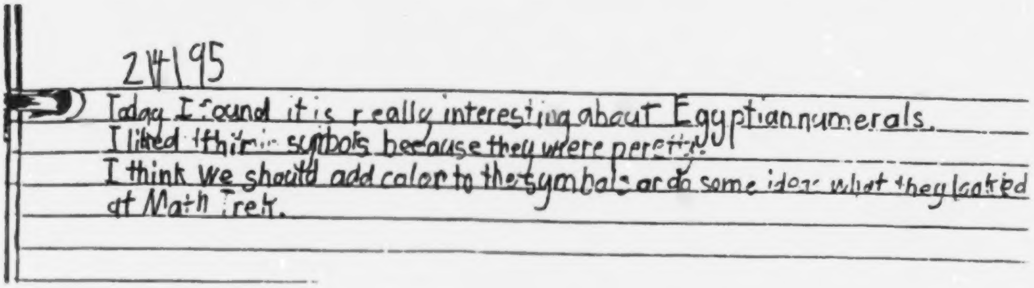
lllnnnn

and waited for the children to respond. They were aghast that the staffs (ones) were placed before the tens and vocalized this by exclaiming, "You can't do that!" The teacher asked, "Why not?" Some of the children seemed quite puzzled and others just shrugged, but no one had an answer or took advantage of freedom from place value on our first exploration of Egyptian numerals, The next week, however, a child put his estimate up on a number line using the following Egyptian numerals: arch staff, arch staff, coiled rope. When asked why he arranged the symbols in that manner he replied, "Because it doesn't matter how you do it in Egyptian—it will always be the same number."

As a wrap-up, children were asked to compare our system with the Egyptian system. The children divided themselves into two camps. There were many children who said they appreciated the Egyptian numerals because they were "pretty" and "arty," not like our numerals. There were also many children who said that it takes too long to write them out; they pointed out that addition and subtraction problems require a lot of counting. We summarized by saying that the group seemed to feel that Egyptian numerals were more aesthetic but more cumbersome.

There are many activities one can do with other numeral systems: play "war" with cards that have numbers written in the numeral system; do estimates on a number line using a different numeral system; do all four operations using them; and compare Roman, Egyptian, Babylonian, Mayan, and Hindu-Arabic systems. Perhaps the unit could end with children inventing their own numeration system.

Coming to learn another numeration system helps children to reflect on their own and to understand it more structurally. In a small way, this is not so different from the effect learning a foreign language has on our understanding of our native language.



2/4/95

Today I found it is really interesting about Egyptian numerals.
I liked their symbols because they were perety.
I think we should add color to the symbols or do some idea what they looked
at Math Trek.

Equivalence

A crucial idea that underlies much of mathematics is *equivalence*. Equivalence involves the understanding that there are alternative but equal ways of expressing values and quantities in mathematics. The most recognizable form of equivalence may well be in place value, where we acknowledge that ten ones is equal to one ten, ten tens are equal to one hundred, and so on. Equivalence, however, permeates all of written arithmetic, including the place value system, equations in algebra, and congruency in geometry. Indeed, can anyone think of anything in mathematics that does not involve some form of equivalence? It is partially by virtue of equivalence, which affords multiple and flexible ways of expressing mathematical ideas and quantities, that mathematics is as rich and interconnected as it is.

Chip-Trading

One of the ways in which we explored the idea of equivalence in Math Trek was through the game of chip-trading (Davidson, Galton, & Fair, 1975). This versatile game is accessible to all children in a class as it can be played at different levels. This latitude to play at different levels and in different ways ensures that all children get something useful out of their play. Indeed, students will gladly think up some of those ways. Thus, this game is also a great example of what we mean by an open-ended activity. The way the game works is this: First, make a playing board by dividing a piece of paper into four columns. Mark the top of the right-hand column yellow, mark the top of the next column blue, the next one green, and the left-hand column red. Then, choose a base starting with a small number such as three (called "Land of Threes"). There are four different color chips (yellow, blue, green, and red) and these represent the first four powers (0, 1, 2, 3) in whatever base is used. Each player takes a turn rolling the dice (or a die in the case of small "lands" such as 3 or 4); the number appearing on the dice tells how many yellows the player is to take. The object of the game is to acquire a red chip. In the Land of Threes, whenever three or more of any color are accumulated, they must be traded in for the next color, as that next color represents the next place value. For instance, let's say the number 4 was rolled. The player would take four yellows and would then need to trade three of those yellows in for a blue, so that the player would have one yellow and one blue on their color-coded playing board. The play continues until someone is able to trade for a red.

Another feature of the game is the use of a "banker" who hands out the chips as requested. The banker's responsibility is not only to hand out the chips, but also to ask for the player's reasoning. For instance, if a child requests a blue and two yellows when she has rolled a five, the banker asks, "How come?" The player then has to articulate her reasoning, e.g., by explaining, "Well, a blue is worth three yellows, and three and two are five." Requesting that children articulate their reasoning accomplishes three goals: a) children clarify and process on a deeper level; b) children expose their peers to more advanced strategies and their rationales; c) teachers can further assess the children's strategies and their understanding of those strategies.

There are also excellent, thought-provoking questions that can be asked as the children are playing, or when one wants to stop the play temporarily to assess who is ahead, by how much, and why. For instance, one can ask: "I wonder how many yellow chips are equal to one green chip?" Children are interested in such questions and willingly disengage from their play to puzzle out the answer with each other. Their answers provide a window onto their thinking about equivalence, groups, and the number of units within a group; concepts that are used in all computations. Some children explain that a green is worth four yellows because a blue is worth three, so a green has to be worth more. Other children will say six, and at least one child will figure out that a green is worth nine yellows because it takes three yellows to make a blue, and three blues to make a green, and three threes are nine. One can then go on to ask what a red is worth. Questions can also be posed about how the value of the green chip relates to the value of the blue chip.

The game can also be played in reverse: All players start with a red with the goal of getting to zero. Going in reverse requires mental flexibility and visualization skills that are very important in higher mathematics. Going in reverse is akin to subtraction, an operation that also seems more difficult for children for similar reasons.

Listening to children's conversation as they play leads to more ideas. For instance, one child giggled, "I know, let's play this game in the Land of Ones!" The teacher responded by saying, "Gee Jeff, you seem to think that would be a very funny idea. How come?" Jeff answered, "If I went first, I'd win on the first throw!" He went on to explain, "Even if you rolled a one, you'd trade that one in for a blue and immediately have to trade that one in for a green and that one in for a red and then you'd win!"

As mentioned in the last chapter, chip-trading is also a good place to work on setting the climate for mathematical exploration. Some children, no matter how bright, like to see all their chips before they trade, or like to take their time to think things out. Other children inevitably want to do it for the seemingly slower child. The teacher's role here is to help children respect one another. One such impatient child pounced on the more contemplative child and said, "Here, let me do it for you. See, you just take a blue and two yellows! I'll get them for you." This of course made the child whose turn it was hopping mad. We intervened by saying, "You know, Richard, Marina has her own way of getting the answer. That's the neat thing about this game; there are lots of ways to get to the answer. I bet you two could learn from each other. Let's watch and see how Marina does it. Then she'll watch and see how you do it." In this way, children learn to listen to and respect each other. They also learn that quickness is not the goal in mathematics. This is also the sort of game that, once children know the few simple rules, they can play by themselves. This frees the teacher to work with another group of children.

After children have played the game in different lands, you can start asking them to compare how it is to play in the different lands—which land is easiest to play in and why, which land is the hardest to play in and why, etc. Getting them to compare different bases invites them to reflect on multiples and factors.

Here's what two children invented after their experience of playing the game periodically over many months: Rachel and Tanya decided to count all the chips of each color and figure out how many points that they amounted to all together in the Land of

Threes. They were quite systematic and purposeful in doing so. They got a piece of paper, cut it into four sections, and labeled each piece with the color of the chip, and how many yellows the green was worth. Then they counted up the chips of each color and did the computation to figure out how much they added up to. For example, they found that there were 35 green chips. They then multiplied 35 by 9, for each green is worth 9 yellows in the Land of Threes. They did this for all the different colored chips and were absolutely delighted with their investigation and excited about sharing their results. We wondered what the results would be in the different lands. They quickly got busy calculating how many yellows each chip was worth in the different lands. We thought the children might be able to appreciate the relationships among the color values if they were presented in a systematic manner. So we asked the girls if they might make a table of the results and then look for some patterns. Systematically recording the results engendered an excited flurry of thinking and pattern-finding. Rachel noticed that the blues were always the number of the land. Tanya noticed that you "times" the land you're in by one and you get the value of the blue chip. Rachel observed that you get the value of the red chip by multiplying the value of the blue chip by the value of the green chip (e.g., 3 x 9 = 27). Then Tanya excitedly chimed in that you "times" the value of the blue chip by itself or by the land you're in (same number) and you get the value of the green chip. We then asked them if they noticed anything about the value of all the greens. They both exclaimed, "Yes! They're the square numbers!" We replied, "Hmmm, perhaps if there is a pattern in the value of the greens, there's a pattern in the reds, too." They seemed stuck at that point so we suggested that we build the value of the chips in the Land of Threes in order to try and see the pattern. Together we used the colored tiles and put down one tile, then three tiles in a row, and then a square of nine tiles. We then asked what the 27 might look like. They did not have any idea, so we suggested we start off with a square of nine. We then asked them to visualize what 27 would look like. Rachel said, "Well, maybe we could have three squares on top of each other because three nines are 27." "Oh!" said Tanya, "It will make a cube!" We then began to wonder if that would hold true for the reds in the other lands. The girls danced around exclaiming that the red chip is always a cube. When it was group time they presented their investigation to the whole class, asking their classmates to find some of the patterns that they found.

Thus, a simple game led to some very complex mathematics. The route to this mathematics was through play and exploration and the problem-posing that inevitably attend the freedom to play and explore.

Visualizing Numbers: Patterns, Functions, Squares, Rectangles, Golden Rectangles, and the Fibonacci Sequence

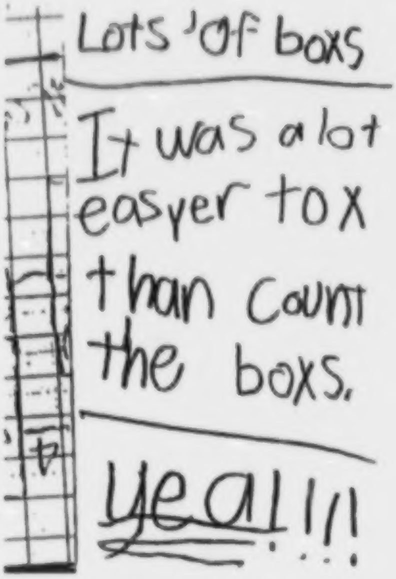
Encouraging children to see the connections between equations or numerical relationships and the images or shapes that those equations describe or engender was an ongoing theme at Math Trek.

The Saturday Club children willingly used manipulatives to represent their thinking and as tools for solving problems. They liked graph paper and were intrigued with the idea of writing number sentences or equations to describe their work with the manipulatives. They also enjoyed going in the other direction, starting with an equation and drawing or representing its meaning with manipulatives. In this way, the children came to see that the two enterprises were connected and both a valid part of mathematics. Visualizing and equation-writing also became two tools in their ever-expanding tool kit.

The children's great pleasure in visualizing numbers convinced us to explore the issue more formally in the second year. We started with an adaptation of Kaye's (1987) "Lots of Boxes" game (from *Games for Math*, p. 123). This game, while simple enough for first graders, opens out into some complex mathematics. It's all in where the children, and their accompanying guides, want to take it.

Lots of Boxes

In this game, children are given a piece of graph paper, a pencil, and one die. The first roll tells the player how many boxes across to draw. The second roll tells the player how many boxes down to go (starting from the end of the first line drawn). With those two pieces of information, the player is able to finish the box by filling in the other two sides to match the lengths already drawn. Of course, some children will insist on throwing the die more than twice. That's okay; they will quickly notice that unless they roll the same two numbers again, they don't get a box. This can lead to a fruitful discussion of why it only takes two rolls to determine all four sides. After the player has completed the box, she is asked to figure out how many squares are inside her box. Then she is asked to write a number sentence that shows how she figured that out. For instance, if she rolled a two and then a four, there will be eight squares inside her box. If she counted them up two at a time, she might write: "2 + 2 + 2 + 2 = 8." Another child might see the relationship of the two lines as having to do with multiplication, therefore writing the equation as: "2 x 4 = 8." Again, this helps the teacher by providing assessment information, and helps the child by relating numbers to an image.



Lots of boxs

It was a lot easyer to x than count the boxs.

yea!!!!

Playing this game leads to an interesting phenomenon. Eventually, a player will roll the same number twice, and notice that the shape that emerges is a square. At this point, teachers can ask, "How come sometimes you get squares?" This question focuses the player's attention on how numbers yield shapes. As Lily said, "Oh, I get it, if you roll the same number twice, you always get a square!" We followed up by asking how come rolling the same number yields a square. Her reply was, "That's because a square has to have the same size sides, and if I roll the same number, that makes it so all the sides are the same."

Then one might ask, "So, are squares rectangles?" thus starting the children off on an exploration of how squares and rectangles relate. This exploration can veer off into two other directions. One is to have the children further explore squares and rectangles by playing the "Martian Game."

In the Martian game, the teacher explains that she is a Martian who has never met a square before and would like to know more about one. However, as a Martian, she cannot see; she can only take the information in through language, therefore the children can't show her squares with pattern blocks or by drawing. The children have to give drawing directions that the teacher must follow literally. The results are often hilarious; children have a hard time articulating their sense of shape precisely and with defining features. This is a good exercise to get them to articulate their mathematical understanding of shape both to others and to themselves. Finally, as a group process, children will eventually succeed in getting the "Martian" to draw the shape accurately. A cheer usually goes up when the Martian finally gets it right. A debriefing discussion can then be had about the importance of being precise.

The "Box" game can also be taken in another direction by having children take note of how many little squares are inside the big square; this can be a good first introduction to square numbers or a new way of seeing square numbers if children have been introduced to them rather rotely. Indeed, one third-grade class playing this game as an introduction to multiplication became quite fascinated with the square numbers and wanted to keep on going by making a list of them. Their teacher wondered aloud if they could find a pattern in the amount by which square numbers go up each time. The children quickly invented the rule that the next square number can be found by adding the next consecutive odd number to the last square (e.g., 0 + 1 = 1, 1 + 3 = 4, 4 + 5 = 9, 9 + 7 = 16, . . .). Their enthusiasm indicated that finding the numerical pattern was a piece of mathematical power for them.

Other Square Activities

Squares can also be further explored through the use of an adaptation of a great problem from Schifter and Fosnot (1993). Farmer Jane has a problem: she loves squares and so has put her modest little apple orchard into the shape of a square. It has done quite well and for next season she wants the next bigger size square. She'll have to add 13 apple trees to get to the next bigger square. How many apple trees are in the current square?

This is another great problem to solve in a group and with some manipulative such as colored tiles or with graph paper. The teacher can then watch to see whether children generalize their knowledge and experience with squares, or start from scratch. Here's what Rachel and Joni did. They started by building a square of four tiles, and then adding on the necessary tiles to build the next larger square. Joni thought it would be a good idea to color coordinate the colored tiles so that it would be easy to see how much they added on each time. This made a pleasing effect—each time the square was made larger, an L-shaped group of like-colored tiles was added. Rachel counted the "L's" and, when the square had 49 tiles, announced that since 13 had been just been added, Farmer Jane's original square orchard consisted of 36 trees. When asked if they could find a pattern in what they were doing and they quickly separated the L's, noting that, "It's like counting by the odd numbers!"

Another follow-up would be to use the Open Court story from *Bargains Galore* that has Mr. Breezy telling a hardware store clerk that he needs eight squares of glass in order to resurface a square window. The children are then asked whether eight squares could be arranged to form a square. Interestingly, the Saturday Clubbers had to work this out by drawing or arranging the tiles in order to see that eight squares cannot be arranged as a square; eight to them was a pleasing number that "seemed right." Approaching the same concept from different angles allows teachers to assess how well children have internalized and generalized the concept, and allows children to understand the concept in both more intensive and extensive applications.

Some children of course seem to come into knowing "all about" square numbers. Beware. Just as with the Mr. Breezy and the square window problem, there's knowing and then there's knowing. A group of children were observing another child making successively larger squares with the pattern blocks. The children decided the largest square was the Daddy square, the second largest the Mommy square, and the little squares were the baby squares. Anna, who had been watching intently but silently, exclaimed, "I just counted the number in each square: Those are the square numbers! I never knew square numbers were square."

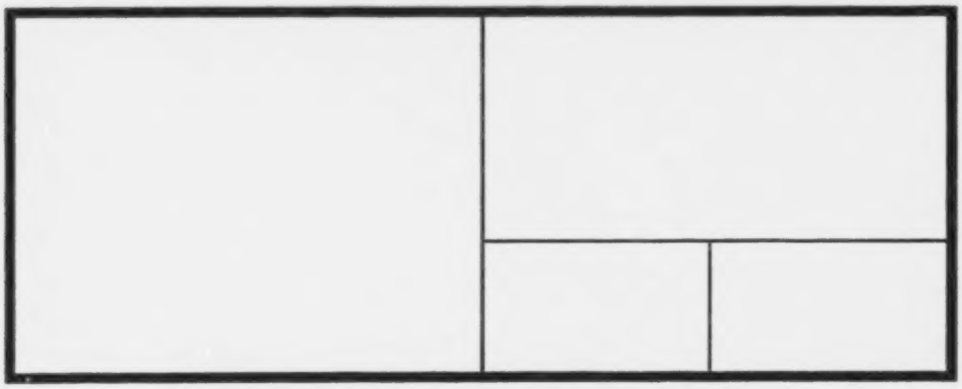
Fibonacci Series

Visualizing numbers (and squares!) also plays a role in any exploration of the *Golden Rectangle* and the *Fibonacci sequence* that describes its properties. This exploration follows well from the other activities described above, and can be conducted in small groups or as a whole class activity. This sequence was discovered by a mathematician named Fibonacci in the 12th century and came about through his observations of how rabbits multiply. This is the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... Notice that the sequence continues by adding the last two numbers. The golden rectangle is any rectangle whose length and width utilize adjacent Fibonacci numbers (e.g., 8 x 13). To learn more about both the sequence and the rectangle, and their roles in art, architecture, and botany, consult any of the following: Doczi (1984); Garland (1987); and J. Gies and F. Gies (1969).

The sequence for this exploration was designed to give children the opportunity to figure out the logic of the Fibonacci sequence and to discern the surprising aesthetic qualities of golden rectangles themselves. Large graph paper for the teacher to use is helpful here, as are individual pieces of graph paper for every child, preferably paper with squares of at least 1 cm. Have on the large graph paper a golden rectangle with the dimensions 21 x 34 or 13 x 21. Ask the children to make their own rectangle with the same dimensions as the one you have. Just making the rectangle involves counting accuracy and may be hard for some children. If this is the case, children could work in pairs, with one child doing the checking and the other child doing the drawing.

The struggle to count off the squares necessary to make the rectangle involves developing the crucial notion that measurement starts from zero. If we look closely at children's measurement mistakes we find that one typical error is to begin the measurement from one. Beginning with one makes intuitive sense, especially from the perspective of a young child accustomed to counting sequences. Recognizing this logical, developmental error can lead to an interesting discussion. Simply being asked to verbalize their intuitions will often help children to become more aware of the issues involved. On a more social plane, it could be a wonderful piece of learning for children to realize that sometimes it is the task, not the teacher, that requires accuracy.

After children have constructed their rectangle (and checked it for accuracy), ask them to figure out the largest square that could be made within the rectangle. Some children will interpret the question as how to make the largest square without touching any of the sides of the rectangle. This is easily clarified by explaining that three sides (or portions of those sides) of the rectangle will also form three sides of the square. After children offer their responses, ask them how they knew that. This question helps children to clarify their thinking about how squares are constructed. Then ask them to find the largest square in the rectangle that's left over, showing them how to move counter-clockwise. Continue to do this with them until there are no squares left. The rectangle should now look like this:

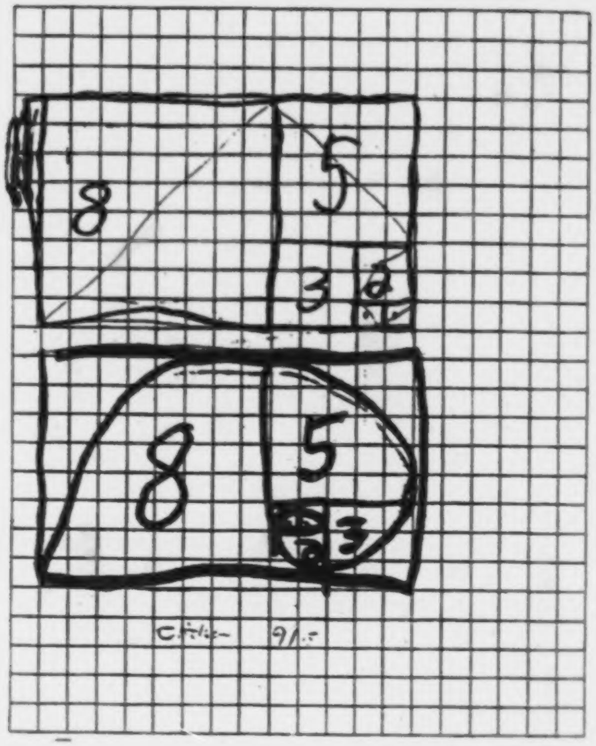


Ask the children how the squares could be described or named. A 13 x 21 rectangle first yields a 13 x 13 square. Some children will say it should be called a "13 x 13," and another child might say that it should be called a "169" since that's how many little squares are inside a 13 x 13 square. The teacher can explain that since it uses the number 13 twice, we could just call it a size 13 square. It helps to write down on the board or overhead all the numbers of the squares and to go in order from smallest to largest. In that case, the size of the squares are: 1, 1, 2, 3, 5, 8, 13. At this point, it can be pointed out that there's a pattern in those numbers and the teacher can wonder aloud, "What would the next number be?" And since children love patterns, they will be quite engaged at figuring this out. They can list their possibilities on the board and be asked to explain their reasoning.

When this was done in the various Saturday Clubs, many of the children noticed that the last two numbers in the sequence always add up to the next number in the sequence. At that point, we informed the children that this was a very special sequence called the *Fibonacci* numbers. We then explained a little about the man who had discovered this sequence in nature—eight centuries ago! Having some books or posters on this sequence in the room will entice the curious kids in the classroom (see Appendix C).

Once having discovered this sequence, many children will want to take it as far as they can go. At Saturday Club, several children figured out all that they could in their heads and then went to find calculators. They spent the next 15 minutes finding larger and larger numbers in the sequence. Two children returned to the next meeting with a page full of Fibonacci numbers that they had worked on at home.

The fun, however, has just begun. After kids have played around with the sequence and generated several more numbers, they are ready to go back to the rectangle to do another activity. If each square is diagonally bisected, starting with the biggest square, a spiral emerges. This spiral elicited appreciative "ahahs" when executed in Saturday Club. It is, indeed, awe-inspiring to see a spiral emerge from a rectangle and the squares within it.



We found that at this point the children we worked with began to pose their own problems and to play with it in their own ways. One child thought that she could draw her own spiral that would be more rounded; free-hand, she drew a lovely spiral on her own rectangle and then began coloring it in a way that honored the sequence. Another girl also decided to color in her square spiral in an aesthetically pleasing way. Other children wanted to make their own rectangles over and over again. These children quickly found out that if you don't cut the squares in order from largest to smallest starting from the left, that the spiral pattern doesn't work. Other children wanted to know if any old rectangle (not with Fibonacci number dimensions) could create a spiral. Their investigations yielded a resounding "No!" For the few children who did not have their own agenda, we posed the following: "What would the next larger rectangle look like? What would the dimensions be, and what would be the size of the squares?" More investigation ensued; the graph paper and pencils went flying. Pretty soon, the children established for themselves that you better use the Fibonacci sequence, and you better take the next number in it (e.g., a 21 x 34 rectangle) and that the squares will repeat except now there will also be a 21 x 21 square. Some children also chose to make smaller rectangles. Andrew wanted to know if you could have a 1 x 2 rectangle and still have it work. He explored that, too.

In this series of activities children were able to see how a shape, in this case, a rectangle, is composed of successively smaller components. They were also able to see, and play around with, the idea that by bisecting each square on the diagonal, a spiral is formed and that spirals of this kind appear in nature as well. This is also a great place to introduce and broaden children's appreciation of mathematical relationships in everyday objects (e.g., pine cones, seashells, sunflowers) and in anatomy, perception, and music. They could also note how important sequence is: cut your squares a different way and the spiral does not emerge. And, they were introduced to a powerful mathematical idea: That a visual pattern can be described by a numerical pattern.

From Number Patterns to Graphing Patterns Via the Ancient Vedic Square

Clearly, playing with squares and number patterns is an ancient pastime: The Vedic Square, based on a transformation of the multiplication tables up through nine, is 3,000 years old. In fact, when introducing this activity to your children, mention how old the square is and ask them to calculate what date it was 3,000 years ago. When we posed this question at Saturday Club, an interesting debate occurred as to whether or not there could be negative years. We found that interjecting a problem to solve as we're busy setting up an activity helps to focus the children. If it's an interesting problem for the children, they will continue to think about it for a while.

With the children, we constructed the multiplication table from one through nine, leaving a space between each line of the table. Alternatively, you could simply present this table to the children on an overhead. However, that would deny children valuable multiplication fact practice. By embedding fact practice within an activity, all children are more willing to work hard to secure the answers than when the answers are the goal in itself. And, for first, second, and early third graders, these "facts" are genuine problems to be solved. Toward the latter half of the table, some children might get bogged down in their calculations; calculators come in handy at this point.

Once the table is constructed, we announced that the ancient Indians did not want two-digit numbers in their table; they would only allow one-digit numbers. We then posed the problem: How could we convert the two-digit numbers into one-digit numbers? The children offered several solutions: "Just drop one of the numbers!" "Pick a number in between." "Wait, that can't work if you have a number like 10. Then you'd have a fraction." "I know—add the two numbers together." At that point, we said that the last suggestion was exactly what the ancient Indians came up with. We then proceeded to put all the one-digit numbers underneath the original line in the table. When we got to a two-digit number, we added the individual digits together and went on. Here's what the table looks like as it is constructed from the multiplication table:

1	2	3	4	5	6	7	8	9
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	4	6	8	10	12	14	16	18
2	**4**	**6**	**8**	**1**	**3**	**5**	**7**	**9**
3	6	9	12	15	18	21	24	27
3	**6**	**9**	**3**	**6**	**9**	**3**	**6**	**9**
4	8	12	16	20	24	28	32	36
4	**8**	**3**	**7**	**2**	**6**	**1**	**5**	**9**
5	10	15	20	25	30	35	40	45
5	**1**	**6**	**2**	**7**	**3**	**8**	**4**	**9**
6	12	18	24	30	36	42	48	54
6	**3**	**9**	**6**	**3**	**9**	**6**	**3**	**9**
7	14	21	28	35	42	49	56	63
7	**5**	**3**	**1**	**8**	**6**	**4**	**2**	**9**
8	16	24	32	40	48	56	64	72
8	**7**	**6**	**5**	**4**	**3**	**2**	**1**	**9**
9	18	27	36	45	54	63	72	81
9	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**

When we arrived at 28 (4 x 7), the children immediately noticed that when you add the two digits, you still have a two-digit number. We asked them what they thought we should do about that. Two children quickly chimed in: "Add those two numbers together, too!" Indeed, that was the solution the ancient Indians used. Thus, 28 becomes 2 + 8 which adds to 10; 1 + 0 equals 1.

Here is the actual Vedic Square, without the original multiplication that it transforms:

1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

After the table was completed, Cindy announced that the table was not really a square. We asked why, and she replied that it didn't look like a square. At this point, we erased the original lines of the multiplication table. It would have been helpful to have had an overhead of just the Vedic Square in square shape (as above), but we worked with what we did have. We then posed the idea that perhaps if it didn't look like a square, that could be due to the way in which it was drawn, and asked, "Why do you think the ancient Vedic people considered it a square?" The children needed to think for a while. Jon said it absolutely was not a square; it did not have a square shape. Sean said, "But it has to be; it's a nine by nine. If you count down and then across, it's the same number." Tanya said there sure were a lot of nines in this square. Lainie exclaimed, "Why didn't they just call it the magic nine table?" We interjected at this point to say, "Let's explore this table to see what other patterns we might see."

Thus began a veritable explosion of ideas. Patterns were found on the horizontal, vertical, and diagonal planes and squares within squares were detected. Much excitement was generated by this search for patterns, and the children delighted in coming up to the blackboard and pointing their pattern out to the group. There was some repetition in the patterns that were discerned. We handled this by saying, "Oh, you saw the same thing that Paul did." We also commented, "Isn't it interesting how a pattern can be pointed out, but we sometimes need to discover it for ourselves?" In this way, children did not feel awkward for not listening, or perhaps simply not seeing what others pointed out.

After about 15 minutes of pattern-finding, the children were quite giddy with all the possibilities. We began to concentrate on the patterns that were reciprocals of each other (i.e., the complements of nine such as 1 and 8, 2 and 7, 3 and 6, 4 and 5). The children began to get restless at this point; a good indication that we were pushing a bit too hard here. We backed off, making a mental note to come back to this, and then posed the following question: "Where are the patterns? Are they in the square, or in our heads?" This question rather shocked and intrigued the children; it seems that no one had ever asked them to think about such a question. After a pause for reflection, several children raised their hand. Sean said the patterns were definitely in the square because that's where you find them. Amber said that she thought they must be in our heads because we had to use our brains to come up with the patterns and to "see" them. Paul, who seemed delighted with the question, announced: "I think it's a combination. I think it's both in our heads and in the square. And when our brains see the numbers up there like that, they think it's a pattern. So it's both. In our heads and in the square."

And so our first session with the Vedic Square ended with an epistemological discussion about where knowledge exists. With such heady stuff in mind, the children ran to the gym for recess.

Graphing Vedic Square Patterns

The next time the children came to Saturday Club we had the Vedic Square up on the blackboard again. The children enjoyed spotting new patterns and reviewing the patterns they had found before. We asked them to particularly notice the reverse patterns they had spotted before. Looking across the rows or down the columns one finds that one and eight, two and seven, three and six, four and five are all connected by this idea of a reverse pattern. Once these pairs were established, we asked them what they might notice about the four pairs of numbers. It took awhile, but finally Sean exclaimed, "They all add up to nine!"

We then asked, "What is it about this table? There seems to be so much that has to do with nine!" The children agreed and decided it should be called "The Magic Nine Table." Paul wanted to know if the ancient Vedic people used this table for anything. Tanya wanted to know what is it about nines that leads to so many patterns. Lainie chimed in with her trick for multiplying with nines, and there was a general discussion about nines.

The next step was an invitation to graph their favorite row from the Vedic Square (e.g., if you choose the first row, you will be graphing all the numbers in that row: 1, 2, 3, 4, 5, 6, 7, 8, 9). We explained that they could take a sheet of graph paper and pick a place to start, preferably in the middle of the sheet. The first direction was to take the first number in their row and draw a line as long as the number of boxes that corresponds to that number. The second direction was to take the second number in the pattern and to continue drawing a line, this time going down on the graph paper. For the third number, the child was instructed to continue to the right, and so on, always turning to the right for the next number until the pattern is completed by a return to the point at which the graphing started.

The children assiduously worked on this task for about 20 minutes, every now and then exclaiming with surprise as the row they were working on emerged into a visual pattern on the graph paper. Lauren was dismayed to find that she had run out of room on her sheet; she quickly jumped up, grabbed another sheet and some scotch tape and created a larger sheet on which to work. After they were finished with their graphing they spontaneously started to name or describe their patterns: "Oh, this one looks like a windmill." "Mine looks like a kite flying!" "Mine never returns to where I started, it just keeps going on and on."

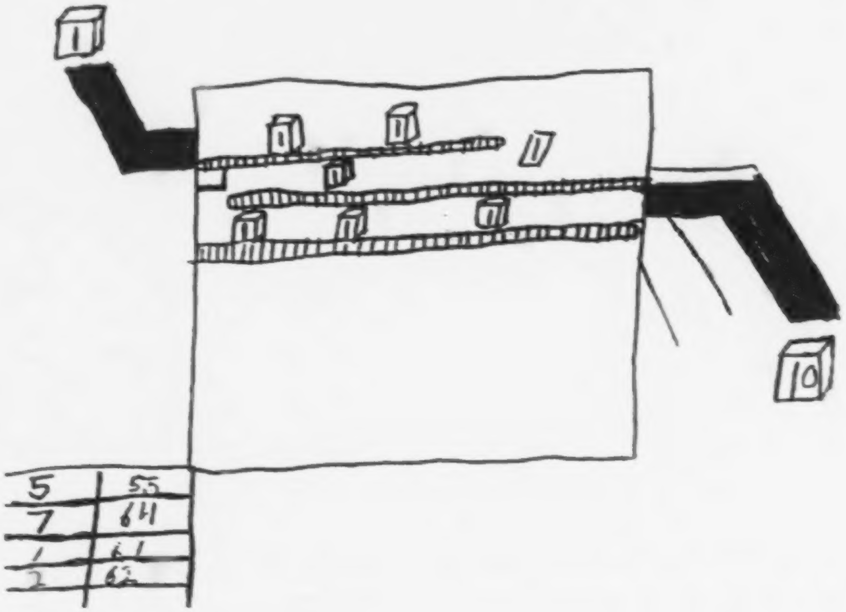
We hung the various graphs up on the wall to make a gallery of graphing patterns. The children oohed and ahhed over the various patterns. We then paired up the reverse patterns (the complements of nine). The children were charmed to see that the numerical reversals worked as visual reversals. Such correspondences between the numerical and visual worlds seem to create not only delight for these children, but a sort of security, too.

This graphing activity allows children to explore the rich and somewhat surprising idea that a number pattern can also create a visual pattern. In this case, children have gone from the numerical to the visual; the opposite path from the one we took with the Golden Rectangle. Going back and forth from the numerical to the visual, and the visual to the numerical facilitates for children a deep appreciation for the ways in which the mathematical language of numbers is also a descriptive language of shape and form. Going back and forth from the visual to the numerical also highlights that one can start at either place (number or shape) and get to the other, another example of reversibility. Typically, children do not have access to this big idea until sometime in secondary school. Why withhold one of the great aesthetic uses of mathematics and its purpose as a descriptive language until so late in a child's life?

Visualization of Function: Magic Number Machines

Another way to begin play with the visualization of number is through imaginary magic number machines. This game uses simple functions in a "Guess My Rule" format. For instance, when the number two is put into the magic number machine and a 10 comes out, then a three is put in and a 15 comes out, the rule must be "multiply by five." A graph with X and Y axis can be drawn, and the numbers that are put in and the numbers that come out (e.g., 2 and 10, 3 and 15) become coordinates that can be graphed as points. Line segments, through the points on the graph, can then be drawn. This game provides a simple but elegant introduction to coordinate graphing and enables the children to see that they can predict from the graph, number relationships that they had not yet graphed ("Cool!" was Maggie's word for this discovery). Children enjoy making up the rules for parents to guess as well as the reverse. As children become more experienced with this kind of graphing, squared and cubed functions can be introduced as well as negative numbers.

In one class, work with the magic number machine took a playful turn worthy of Lewis Carroll. Alice decided to put a sweater with three buttons into the imaginary machine. This time, the machine was still operating on a "multiply by five" rule. The children were quick to wonder if it would come out as five sweaters with three buttons each or five sweaters with 15 buttons each.



Everything Can Be Measured

Measuring things seems to have the same innate appeal and satisfaction to young children as naming things; it's a way of coming to know an object. There are several big ideas that relate to the topic of measurement. One big idea is the notion that the unit of measurement needs to be a standardized unit. A simple exploration of what happens when children measure the length of a rug using their own feet and then compare the outcomes can quickly alert children to the necessity for a standardized unit. Another big measurement idea is embedded in measuring with conventional units of measurement such as inches and feet or pints and quarts. Conventional units of measurement such as inches and feet or pints and quarts pose the same cognitive challenge as time does; children need to construct the concept of how units relate proportionally to each other (e.g., there are 12 inches to a foot). Children first grapple with measurement in ways that are similar to how they first encounter the measurement of time, through qualitative assessment which involves using any material at hand and then simply counting up how much of the material was used. Of course, this method can lead to certain problems, which, in turn, can create the cognitive conflict that is so often the impetus for cognitive growth.

For example, in one early Math Trek session some of the participants, having finished their job cards, spotted the tub of Unifix cubes. Kelly exclaimed, "I know! Let's measure how long this room is!" Several children threw themselves into this task, assiduously connected Unifix cubes, and laid them out in a straight row from one end of the classroom to the other. After they were done, they sat back and gazed at their work in a satisfied way until Mike pointed out that they still didn't know how many Unifix cubes they had used. Jenny said, "That's not a problem, we'll just count them all up." So they began to count, one child starting at one end and another child starting at the other end, and yet another child picking a random place in the middle to begin his count. It was "counting chaos" as they verbalized the counting string in quite audible whispers. The children began to complain that they were getting distracted by the other counters and were losing track. We brought the group together to solve this problem, asking, "What can you do to make the job of counting these cubes easier?" After some initial shrugging of shoulders, Jenny said, "We could have just one person do the counting." Alex objected, "It will take too long. We'll be here all day!" Kelly said, "What if we counted by twos or something?" Jenny, picking up on this notion said, "We could count by fives, that's even quicker!" Mike said, "Tens are even better!" We asked, "How come?" Mike said, "Because they're twice as big as fives!" We then asked the children how they would proceed. Kelly said, "We'll still have to count out each ten before we can count by tens. Let's make a ten stick! Then we won't have to count out 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 all the time. We can just line the stick up, then push it up." The other children were intrigued with this suggestion and crowded around to watch Kelly make her ten stick and then to "measure" with it. The session ended before they could get their final count and they were very distressed. We asked them how they could "save" their work so they could find out the answer the next time they came. Jenny said, "I know! We can ask the kids who come here every day to finish the job. I'll write them a note." Mike said, "Let's write down in our Book of Numbers what we got to, and then put a mark on the rug to show where we left off." We quickly suggested putting down a piece of tape instead of marking up the rug.

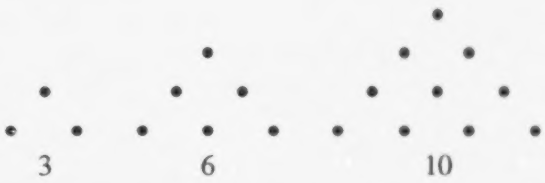
The next session, the children quickly resumed their activity of determining the length of the rug. One child seemed to forget about the use of the ten-stick and where they had left off and began to make another long unifix train. Kelly informed her that she didn't have to do that: "Don't you remember our measuring stick? Let's use that again! And we can start where we left off, so we don't have to make such a long train." The children quickly finished their project, but were dismayed to find that when they came to the end of their measuring, they couldn't put the whole ten-stick down. We asked, "So what are you going to do about the fact that you can't fit another ten-stick down?" Mike said, "Aw, let's just forget about it. We got 27 down, so that's the answer." Alex responded, "You can't do that, you can't just forget about it." Alex then counted how many individual cubes would fit in the space that was left and said, "It's 27 tens and four left over. We can add that on when we figure out how much 27 tens are."

Resolving that last problem might have ended the whole activity except that another group of children became intrigued with the idea of measuring the length of the room. They proceeded in their own way, but came up with a different final count of unifix cubes. The two conflicting answers paved the way for discussion of another important measurement idea: measurement error.

We said at the beginning of this section that children love to measure, and that measurement is another way to come to know an object. Another big idea in measurement is the intriguing notion that anything can be measured. A small group of Math Trek participants practically bumped into this idea as a result of going off on a tangent. Tangents, of course, are themselves mathematical!

From Triangular Numbers to the Measurement of Angles

For the beginning of one session, we put the first triangular numbers on the blackboard. As the children came in, we invited them to come over and help figure out the next number in the series:



At first, the children thought the numbers should just go up by three. Then they stopped looking at the numbers and started to attend to the array of dots formed for each number. Their comments let us know that they were quite tuned in to the patterns the dots made: "I see, it goes up by one row each time." "Well, I think that it goes across one more in each row each time you get to a new number." They then counted up the dots in order to figure out the next number in the pattern.

Rich stood back from the blackboard for a few seconds and exclaimed, "Hey! They're triangles!" All the children agreed with Rich that each of the arrays of dots made a triangle. Then Rich said, "And they're called triangles because they have three angles." "Gee," said Sean, "If a shape with three angles is called a triangle, maybe a shape with just two angles is called a "biangle." "Yeah," said Paul, "And then if it has only one angle it's a uniangle!" We wondered what a biangle and a uniangle looked like. Sean drew one angle and Paul said he thought maybe a uniangle would be a straight line which he then drew. Then he said that a straight line has 180 degrees. The children then had a vociferous disagreement about whether or not a line was a shape or an angle or neither.

We were now far afield from the issue of triangular numbers, but the children clearly were caught up in their discussion of angles. Going with the flow, we interjected the question, "So what is an angle anyway?" Rachel grabbed some chalk and drew an angle and said, "That's an angle." We asked her, as well as the other children, "Well, where do we find angles?" A chorus of answers descended on us: "See that table, that's a right angle." "And the blackboard has right angles, too." Running for the pattern blocks, Paul announced, "And all of these shapes have angles, too."

After several more minutes' worth of discussion about all the places in the world we find angles, we asked, "So how do we measure angles?" The children were silent for a minute and then Rachel said, "I don't know that, but I do know that right angles are 90 degrees." And Paul said, "I already told you a straight line is 180 degrees." We then asked if any of the children would like to explore how to measure angles. A few children were interested while the rest busied themselves with the other activities and job cards that were around the room.

We got out the pattern blocks and asked if they could see any angles. Paul said all the shapes had angles and noticed that some shapes had more than one kind of angle. We then asked, "So, do we know anything about any of these angles that would help us get started?" Rachel said, "Yeah, remember, right angles are 90 degrees." Teacher: "Could knowing that right angles are 90 degrees help us figure out anything about the other angles?"

With that directive, the children became quite absorbed in trying to use their information about right angles to figure out what the other angles measured. Paul took a rhombus angle and placed it on to one of the right angles of the square. He spoke softly to himself: "Hmmm, looks like it would take three of this kind of angle to make a right angle . . . so this angle is about 30 degrees." We asked him how he would prove that. After a minute or so he took three rhombuses and placed an angle from each onto the square. Sean watched all this intently and then picked up a parallelogram and said, pointing to one of the parallelogram's angles, "This looks like two of the diamond's angles, so it would be 60 degrees."

Having found a method, the children worked diligently on finding the measurement of each angle. When Paul measured the larger of the rhombus' angles he said, "Wow! 150 degrees! That's hot!" The other children giggled and commented on how funny it was that the same word meant two such different things. Rich chimed in from another part of the room, "Yeah, that happens a lot with words."

Once the children had "proved" how many degrees were in each angle we asked them to record their findings. We casually placed pattern block templates (stencils that correspond to the pattern block shapes, though children can also trace around the pattern blocks to record their shapes) on the table where they were working. They took a pattern block template and their books of shapes and traced each pattern. They then recorded the number of degrees for each angle right inside the appropriate angle.

When they were finished, we asked them how many degrees there were in each shape. This question led to a hot debate about how many angles were in a hexagon. Paul began to answer the question by saying, "Well, there are six angles in a hexagon, and if each one is 120 degrees, then . . . ." Sean cut him off and said, "But there aren't six angles in a hexagon; there are only three." We asked him what he meant. He took a hexagon and showed me what he considered to be the three angles. The angles he pointed out were not the adjacent ones, but opposite ones. Paul pointed out what he saw as the six angles to Sean, but Sean didn't buy it. We interjected, "OK, you think if an angle shares a line with another angle that it can only count as one angle." Sean agreed, with some measure of relief at being understood. There ensued a hot debate among the children as to what counts as an angle. We asked Sean how many angles he thought were in a triangle. Without any pause, he said, "Three. That's why it's called a triangle." Then he gazed thoughtfully at the little green triangle in front of him and slowly said, "Oh, I see what you mean, I guess it's okay to share lines, it's how many little points there are, that's what tells you how many angles there are."

We then went on to explore how many degrees there were all together in the various pattern block shapes. Paul became the record keeper, using the pattern block template to draw the shape in question and then carefully putting the number of total degrees right in the center of each shape. Helen, a guest aide, and a regular classroom teacher during the week, wanted to know how many degrees there are in a circle. The children joined her in trying to approximate a circle with the pattern blocks. During the process of creating the circle, Rachel pointed out that the shape gets closer to a circle each time more sides are added. (Burns must have been similarly taken with that idea for she has written a book that explores multi-sided shapes called *The Greedy Triangle*, 1994). They went on to figure out the degrees in a straight line by putting two right angles together, which caused Paul's eyes to grow wide for he had already known that "fact" and now he could see why. The children ended their exploration by recording their observations and calculations in their "Book of Shapes." Finally, we asked them to examine their findings to see if they could find any patterns. Sean, who enjoyed systematizing his observations, had written down the measurements for all the angles in descending order. His method allowed him to quickly spot the fact that all the measurements were multiples of thirty. Rachel, who had joined us, softly said, "So **that's** why all those pattern blocks fit together so well!"

This activity can be extended even further by asking children to describe the relationship between the number of sides (or angles) in a shape and the total number of degrees for that shape. The following is an example:

# of sides	**degrees**
3	180
4	360
5	
6	720

Children will notice that they don't know how many degrees are in a 5-sided shape. At this point, they can be asked to construct a pentagon by using two or more of the pattern blocks, and then to figure out how many degrees are in the shape. Alternatively, children could be asked to deduce the number of degrees by examining the data already on the table. After the pentagon question is solved, the children can be asked to figure out how many degrees in a 12-sided shape. Some children may even be interested in creating an equation that would show how to derive the number of degrees by knowing the number of sides.

Estimation

Another ongoing theme over the two years of Math Trek was *estimation*. Estimation is a versatile mathematical tool that introduces children to the importance of flexibility in mathematics. Playing with estimation inevitably leads to questions: When is it reasonable to estimate? What is a reasonable estimate? When is it important to be precise?

At Math Trek we played with estimation in a number of ways. One of the teachers made estimation a running theme in her classroom by instituting an estimation jar. She further personalized it by asking the children to take turns bringing in something to estimate each week. The children were very invested in this activity, taking great pride in coming up with an estimation jar when it was their turn. Jars came in filled with rocks, cookies, popcorn, beans, macaroni, and, most ingeniously, with broken egg shells. It was the week after Easter, and all those dyed eggs were put to a final good use. The children were charmed and intrigued by this jar of crumpled egg shells and became engaged in a heated debate about how many broken egg shells would constitute one whole egg shell. Thus, estimation activities may lead to considerations of area, volume, and other mathematical topics.

The teacher in this class also devised a unique way to elicit risk-taking with estimation activities. Often, when asked to verbalize their estimates, there is a distinct tendency for children to play it safe and give an estimate close to the one they have just heard. Here, however, the children were asked to write their estimates simultaneously on a sticky note. This way, the children were not aware of what anybody else was estimating, and were free to figure something out for themselves.

After their estimates were recorded, the children were invited to go up to the blackboard and place their estimates along an imaginary number line. That is, the children were asked to order and sequence their estimates without any markers. In this way, the task became more challenging and involved other aspects of number sense. Conceptualizing numbers in this spatial way also encouraged the children to see numbers in relation to one another. If Janie just put up her estimate of 100 smack in the middle of the blackboard, where will Reggie put his estimate of 125? At the end of the board or closer to the 100? Well, it would depend on where Tim had put his estimate of 200. Statistical concepts of range, mode, median, and mean become visual, accessible, and meaningful when put in this context. These statistical concepts can also be highlighted in a discussion of the children's estimates after everyone has placed their estimates on the board. Questions such as, What's the smallest estimate? What's the largest estimate? What's the difference between them? What's the most popular guess? What's the average guess? What's the median (middle) guess? help children to focus on ways to make sense of data. Asking children to provide a rationale for their guesses encourages them to be reflective and alerts other children to alternative strategies and dimensions.

Estimation can also be woven into the fabric of school life in casual ways. For instance, snack time at Math Trek often provided informal estimation exercises that seemed natural and reasonable to the children. The dried lemonade, the jars of cookies, the packages of pretzels were all fair game for initial estimates. The children were then asked to quantify the amount of snack they were given, and to revise their estimates of how many items had been contained in the jar or package. Their strategies included counting the children present and multiplying that number by the number of cookies or pretzels they had in hand. Some children used serving size and weight information on package labels in their calculations. Subsequent revisions were often closer to the mark, and the children were packing away more than calories with their snacks.

The *Open-Court Real Math Thinking Stories* has several engaging stories concerning estimation. The character named Ferdie loves to estimate and finds that most mathematical situations in his life require only estimation. Portia, on the other hand, thinks that precise answers are required. Naturally, Ferdie and Portia come to loggerheads on this issue. The plots involve them in situations favoring one or the other. The children, at first drawn in by the humor in the stories, also began to identify with one or the other of the characters. Johnnie fondly began to refer to the characters as "Estimating Ferdie and Precise Portia." The children easily generated descriptions of comparable situations in their own lives.

Probability

Some big ideas were not so much thematic units as they were *leitmotifs* that were returned to again and again. Probability was one of them. The topic of probability is one that permeates our lives, from the time we are in our cribs trying to figure out what will make those grown-ups come and get us, to our first board games with dice, to our grown-up concerns about earthquakes, disasters, insurance, and lotteries.

This is also a topic about which even young children develop some working intuitions. For instance, in playing the game "Lots of Boxes," the children noticed that it was easier to roll two different numbers on two throws of the die than it was to roll the same number twice in a row. Then there's the old coin toss trick: Will the coin come up heads more times or tails more times? Suppose you come up with heads ten times? Does that mean you'll be sure to get tails next time? Most children, indeed, most adults, will venture that the coin is sure to be tails next time. The idea that each toss of the coin is an independent event is a difficult concept to master, and rather counter-intuitive. Indeed, it takes many years to come fully to believe and understand that each throw of the die is independent. Probability is one area where magical thinking still has a stronghold even on adults' reasoning. Thus, at Math Trek, we decided to make sure that the children encountered and worked with notions of probability throughout the two years, taking particular care to exploit those times when issues of probability surfaced naturally, as with the "Lots of Boxes" game or with the game of "Pig" (see below).

We also tried to elicit and stimulate their thinking about probability by reading various stories that embed probability issues, and playing games that specifically involve the concept. The mathematics book in the Childcraft series entitled *Mathemagic* includes some rich play with probability that is easy to act out in the classroom. In one story, Prince Ali Kwazoor makes a dangerous journey to find the Treasure of Samarkand. He finds the treasure, but before he is allowed to have it, the Wizard of Hind informs him that he must pass a test. The Wizard holds out two boxes. In the red box, there is one black pebble and three white ones and in the yellow box there are seven pebbles, three black and four white. The task is to pick a black pebble from one of the boxes without looking.

In one class, the children were very excited by this story. The teacher had her boxes ready, and each child got to pretend to be Prince Ali making his choice. Of course, unlike the Prince Ali of the story, these princes had to explain their reasoning, too. Most children picked the yellow box, explaining that having three black pebbles gives you more chances to pick a black than the red box which only gives you one chance to pick a black. One child thought perhaps the red box would be better because there were fewer pebbles to choose from. No one realized, however, that 3/7 was bigger than 1/4.

A similar story is told in the *Real Math Series* (see Annotated Bibliography). This story, entitled *Iron and Gold*, embeds a series of problems involving probabilities as children choose bags containing the most gold or rubies. These stories are valuable because they pose the same issues using different quantities, thus stretching children's thinking about probability as well as the role of quantity in probability.

Another game from the *Mathemagic* book involved figuring out your chances of rolling a particular sum when using two dice by listing all the combinations that could get you to that sum. This exercise was handy for helping children realize the importance of writing down results in order to keep track of things. The follow-up activity after figuring out all the combinations for each of the possible sums one gets by rolling two dice (i.e., 2 - 12) was to roll a pair of dice 50 times, keeping track of the numbers that came up. This exercise demonstrated that if there are more ways to get to a sum, then that sum will be rolled more often. As Sarah exclaimed, "Would you look at that? Six, seven and eight do come up more often! And now I know why."

Another way to involve children in thinking about probability is to play the game of Pig. Choosing either addition or multiplication, one person (preferably the teacher) rolls two dice and calls out the number. For addition, the numbers that appear on the dice are added together; for multiplication, the numbers are multiplied. The object of the game is to attain the highest possible number before a one is rolled. As long as neither die rolled turns up a one, the players are allowed to put the combined number down on their sheet. They can then choose to continue the play or to stop there. If they continue to play the game, there is always a chance that a one will appear on the next roll. If a one does appear, they lose all the points they have already accumulated. Children love the risk-taking involved in this game and are unwittingly gaining practice with addition or multiplication (or both, as they must tally up their score to see who won).

Inevitably, the discussion that occurs during this game will begin to focus on the issue of when a one will be rolled. As Kenny said, "I just know it's gonna be a one next time, I just know it." "Well Kenny, how do you know that?" " 'Cause there hasn't been a one yet and we're on our seventh round." "So, if you haven't rolled a one six times in a row, it has to be a one the next time? It sounds like you have a rule in mind: Ones always turn up if you roll the dice seven times. Does everyone agree with that rule?" Andy frowned a bit and shook his head. "Andy, you look like you disagree." Andy slowly spoke up: "I don't think so. I don't think we can tell if a one is going to come up next. There isn't any way to tell."

We asked the children to predict what would occur on the seventh roll. The children quickly chose sides, with exclamations of, "It has to be a one next! It has to be!" or "Nah, it could be any number." The dice were rolled ceremoniously. A two and a six came up. Kenny shook his head in disgust. We said, "Hmmm, what does this mean that a one still didn't come up?" The children shrugged and thought for a while. Laura piped up, "That you can roll and roll the dice and you never can tell what it will be?" We asked, "So even if these dice were rolled a hundred times we might not roll a single one?" Kenny spoke up again: "No, we would have to get a one some time. Maybe on the eighth or ninth roll."

The children still seemed perplexed, so we dropped the issue and continued to play the game. As it turned out, on the ninth roll, a one did turn up, and all those still in the game had to forfeit their points. We asked if they had a new rule to determine when a one would come up. Rich said, "You rolled a one the ninth time, so it takes nine turns!" Sarah said, "But it could come up any time in those nine rolls."

It was time for recess, so we didn't pursue the topic any further. Even if it hadn't been recess, it was time to end the discussion for the children were feeling a bit rattled. However, a germ of an idea had been planted that would have to coexist with or supplant their intuitions about probability. A good way to invite children to test their hypotheses about probability would be to graph the numbers rolled whenever the game of Pig is played. After several of these graphs were constructed, the issues surrounding the rolling of a one might become clearer. Again, the idea that each and every roll of the die is an independent event, with the same probability of rolling each number represented on the die, is a developmental concept that may well take years to grasp fully.

Sampling

Also relevant to both probability and statistics is the important idea of sampling: How can we get information about a population by taking samples of that population? This subject is accessible even for young children to contemplate, especially if the sampling issue is relevant to their lives. The Exploratory Data Workshop at the University of Washington exploited one such relevant case when the company making M&M® candies replaced tan M&M's® with blue ones. Children were given covered dixie cups with 12 candies in each. The task was to determine whether the candy came from a new bag or from an old bag. After poking a small hole in the cover, the children shook out one candy and were asked to put a sticky note on their cup that announced whether they thought their candy came from a new bag, an old bag, or if the data were inconclusive. The results were graphed. They then put the candy back, shook the cup, spilled two candies, and repeated the decision-making and graphing process. The graphs revealed how many more children were correct when using the larger sample. Thus, children confronted sampling variability and how sampling size affects judgment.

Estimation activities can also be a good lead-in to sampling. In one class, a jar of multicolored beans were brought in for the children to estimate. After the children made their guesses, several children were eager to determine the actual count. Given the fact that there were over 2,000 beans, this was a time-consuming activity. The children employed a variety of strategies, including counting by tens, counting by twos, and counting by fifties. Two enterprising children found a paper cup, filled it with beans and then counted the number in the cup. They then ascertained how many cups the jar could contain. Their final step was to multiply the number of cups by the number in the one cup that they had counted. Their answer was quite close to the other counts. Their strategy led them to another problem: They wanted to know how many of each kind of bean there were. Janie and Allen could see that there were four kinds. Janie thought that they were evenly distributed and that they could just divide the total number of beans by four. Allen was not convinced. He thought that there were more red beans. We asked the children how they could resolve this dilemma. Sonia said, "Oh, no, I don't want to do any more counting!" Janie piped up, "We don't have to count the whole jar again! We could just count how many of each kind of bean in one cup. That would tell us." Allen was doubtful: "But maybe not as many reds will get into that cup. How can you tell if what's in that cup is the same as what's in the jar?" Sonia said, "I know. We'll count more than one cupful of beans. Then we can compare results!" Allen, his brow wrinkled, said, "If two cups are good to count, wouldn't three be even better?"

So the children broke up into three groups and diligently worked on this new bean problem. We asked them to record their results in table form so it would be easier to read. The children were amazed by their findings. Not only were there different numbers of the different colored beans, but the total number of beans varied slightly from cup to cup! In our group discussion concerning the results, feelings ran high: "How can we ever know what the truth really is if we have all these different answers?" This was a great opportunity to introduce the idea of sampling error. We also wondered aloud how we could deal with the different numbers on the table. Craig offered, "Find out the average!" This suggestion seemed to satisfy the children.

Conclusion

There are, no doubt, many other Big Ideas and Big Questions that would engage children in making sense of mathematics, as well as many other activities that would also provide good paths into the Big Ideas. By keeping our attention focused on both Big Ideas and children's thinking about those ideas, curriculum planning was consistently a stimulating, reflective, and educational process. It was also an ongoing process; if an activity or question did not work out well, our own learning and mucking around with concepts and activities gave us the flexibility and knowledge to try another tack.

Other Big Ideas that we interwove into our curriculum included the idea that there is a history to mathematics and that people invented it; that numeric systems involve different bases (e.g., the Babylonian system is based on 6); that there are different kinds of numbers (e.g., negative, rational, irrational); and the idea of reference units (e.g., in fractions, any number, such as 12, can be the whole as in 2/3 of 12). In the next chapter, we describe ways in which to integrate math into other discipline areas (and vice versa) and we also take up the issue of assessment.

CHAPTER 7: Integrating and Assessing Math

On the way home after the first session of the second year of Saturday Club, Peter's mother asked him what math he had worked with that day. Peter replied, "It was fun, but we didn't do any math." Switch now to the end of this second year when the third round of testing began. Peter was tested and on his way home his mother said, "Remember last year when you were tested and the tester asked if you did any math at home and you said we never did any math at home?" Peter replied, "Yeah, but that was before I knew math was so many things!"

Peter had tapped in to a major goal of Saturday Club: To help children to see the numerous ways mathematics permeates their world. To that end, many of the curriculum activities were designed to broaden and expand children's awareness of mathematics and its many applications in the world. One method we used to accomplish that goal was to integrate mathematical topics with other disciplines such as science, art, history, and literature. Thus, when we explored different numeration systems, we presented them in historical context, discussing ways in which the symbols might have something to do with the lives of the people who invented that system. Art was an integral part of Saturday Clubs, from origami to making colorful designs with Golden Rectangles, to pattern-making with pattern blocks and inch cubes as well as tessellating and tiling. A tie-in to science occurred with our exploration of batteries and bulbs. For literature and language arts we made extensive use of children's books, stories that incorporated mathematics (e.g., *Open Court* and the *Anno* books) and assignments that invited children to write their own stories for various types of equations. We now explore these interrelationships.

Math and Literature

Given children's love of stories, bringing books into mathematics and mathematics into books is a natural and rewarding combination. There are many "How-To's" on the market now that encourage teachers to use children's literature in the math area (see Annotated Bibliography). These books describe extensive units that tie in to specific works of children's literature that can be used to teach specific mathematical concepts and skills such as probability, graphing, and measurement. We would caution, however, that such tie-ins be pursued with a light touch; both mathematics and literature are to be prized for their inherent aesthetic and intellectual qualities and neither subject should be forced to be a hand-maiden to the other.

However, having a light touch also involves making the most of opportunities that come up almost naturally. Perceiving these natural opportunities involves teachers themselves seeing how much mathematics permeates the world and then creating a climate in the classroom where no matter what is being studied, mathematics can be brought in where appropriate.

For instance, in the first year of Saturday Club, telling stories and reading stories were good ways to help children focus and to build group cohesion. One class of children thought they were too big for picture books and resented having any read to them; for this group we told stories instead, embedding an interesting mathematical problem within the story. In one example, a strong visual image was created as a context for the popular chicken and cow leg problem. In this problem, a farmer sees 24 legs through a fence; how many belong to cows and how many to chickens? A story was created about a person walking down a long dusty road in Tanzania, where the red clay in the Great Rift Valley colors the road red and creates a lot of dust. In the distance, 24 legs are perceived behind a fence, but the cloud of dust makes it impossible to determine how many of those legs belong to ostriches and how many to warthogs.

The children were spellbound during the story and moved off soundlessly to figure out the answer, together or alone, using drawings, unifix cubes, or inch cubes. Thus, this story respected their "maturity," won their attention, and got them engaged in a problem they might otherwise have scoffed at. At the end of the work time, the children were also quite willing to share their different strategies and solutions and to marvel together over the fact that there was more than one right answer.

For another group of children, who loved picture books, a variety of materials were used. In one instance, the delightful book, *A House is a House for Me* by Hoberman (1978), was read to the children. They loved the rhyming story, the ingenious and detailed pictures, and the unexpected notion that almost any object can be viewed as a house for another object. Extending the metaphorical use of "house," the children came up with a profusion of examples of their own. We suggested that they also look around the room. Pretty soon, the examples proliferated, yielding such charms as: a hard drive is a house for software, a desk is a house for pencils and paper, and a blackboard is a house for writing. We then wondered aloud about what the various math manipulatives might be "houses" for. The children erupted with: "I know, I know, pattern blocks are houses for patterns, base-10 blocks are houses for tens, and tangrams are houses for shapes!" In this imaginative, humorous way, mathematical ideas were effortlessly tied into a story. And children were invited to conceptualize the meaning of manipulatives.

Some works of children's literature are designed with mathematical concepts in mind. One glorious example is Anno's *Mysterious Multiplying Jar*. In this gorgeously illustrated book on factorials, children are invited to contemplate numbers that increase exponentially. For the first several pages, children can be invited to do the calculations mentally; after that, they need calculators. Large numbers have a special fascination for young children, and this is in evidence on their faces as they watch the numbers grow ever larger. As Cathy said, "Ouch, my head is beginning to hurt with all those big numbers!" The concept of factorials and the way in which factorials make numbers grow so large proves to have staying power for children. Weeks after this book was read, children would mention it. And months later when we constructed an alphabet of mathematical terms, factorial was mentioned for 'F' words.

Other Anno books were also tremendously popular in Saturday Clubs. In the first volume of *Anno's Math Games*, combinatorial logic was introduced through the issue of inventions that result from combining other inventions. For instance, think of the common pencil with an eraser, alarm clocks, and umbrellas with handles. The children were assigned the task of coming up with their own combined inventions and drawing them. In the same book, the concept of coordinates was explored through creating "addresses" for seats in a movie theater. This story could be used to great advantage as a prelude to a study of coordinate graphing.

Yet another Anno book, *Anno's Hats*, with its probability play, was read to each group of Saturday Clubbers. In one class, however, children took matters into their own hands. They had listened patiently to the story and struggled with predicting which hat was on which character, but they hadn't been as excited or as engaged as the teacher had hoped. Just as the story was finished being read, a postal worker came to deliver some mail. Julia looked up, scurried to get a piece of paper and a pen, and quickly scribbled a message. When the teacher took the stack of mail from the postal worker, Julia ran over and put her message on top. Then she said, "Joy, please read that letter to us!" Joy was bewildered, but complied. The note said: "Please come to a performance of *Anno's Hat* right after recess."

During recess Julia corralled a couple of her friends and they made hats in the colors described in the book. They quickly rehearsed and were ready for their peers when they trooped in after recess. They then proceeded to dramatize the book. The other children were enthralled and participated far more than when the book had simply been read to them. Having the book enacted also helped them to focus on the issue of probability.

In this example, we see that when children are comfortable they can take the initiative and find a kinesthetic way to take charge of material and make it their own. Children's spontaneous acts often pave the way to "teachable moments."

Math and Science

From astronomy to the chemistry of cooking, children's early appreciation of science requires the use of math to quantify, compare, and classify. Experience with the uses of mathematics in science, and the many scientific contexts that use math as a tool, will help children to understand different types of numbers and to create an internal number line, from negative numbers (say, a centigrade thermometer) through very small fractions (say, 1/8 of a teaspoon) to the headiest of large numbers used to describe the space between our planet and a distant galaxy. Math, in the context of science, illustrates the interplay of processes such as measurement, estimation, and calculation, and the necessity for a logical system of written representation that can express the very small and the very large as well as how to arrive at those quantities. Thus, scientific activities, experiments, and thinking can help children see that aspects of math that are usually studied in isolation all have a place and a purpose.

While even some math-talented children are mystified by conventional ways of representing number, other children are fascinated by the logic that underlies numerical representation, for example, using positive exponents and powers to describe astronomical distances and negative exponents and powers to describe atomic weights. Astronomy and chemistry provide two incentives for grappling with these kinds of numbers and their written expression. Considering two areas together may well help the children who are mystified as well as intrigue the children who easily grasp the logic.

One way into science is to begin either by asking children big questions about science or listening carefully for children's own question posing. These big questions elicit children's penchant for theorizing about a phenomenon, and their nascent ability to theorize collaboratively. Wonderful examples of this approach to science are described in the book, *Talking Their Way Into Science: Hearing Children's Questions and Theories, Responding with Curricula* by Gallas (1995), a first-second grade teacher from Brookline, MA (see Annotated Bibliography).

All children's (and adults') theories revolve around a mental model of what a phenomenon is and how it works. These mental models may start off as inarticulate and vaguely formed, but will become elaborated when focused by essential questions and opportunities to explore the phenomenon and make it "work." The following is a description of one of Math Trek's scientific explorations. What were the connections between this exploration and math? While there were no explicit connections, there were mathematical ideas such as proportional reasoning, the idea of limits, and equality of intensity that underlie the investigation.

Mental Model Building

During both years of Math Trek we wanted the children to experience some scientific activities that would inspire them to think about cause and effect, to theorize and hypothesize, and to invent their own experiments. One set of materials we used were batteries, bulbs, and wires. These simple materials were enough to generate several sessions worth of intense exploration and plain fun. In many ways, it was the optimal science project for primary age children because it deals with a phenomenon all children have encountered and about which they are curious. The project utilized easily accessible materials that were easy to work with and, on the intellectual plane, provided a hands-on way to think about causality. Best of all, the materials extended a wonderful invitation to play and to experiment and to wonder about how something works.

The first year, we gave each child a battery, a bulb, and a wire, and simply asked that they find a way to light the bulb. Observing the various ways in which the children approached the task gave us more information about individual styles and tempos. When the teachers discussed the session afterwards, it was apparent that the different classes had developed distinct personalities. For instance, in one class the children were silently industrious. In another class, the children were excited and spontaneously began talking about electricity, using words and phrases that they had previously heard or been "taught." Despite their verbal knowledge of electricity, however, most children had great difficulty making the bulb light up. Some children seemed to be approaching the task quite randomly, while others seemed to be engaged in systematic trial and error. Some thought that all they had to do was to put the bulb on top of the battery in an attempt to copy the evident design of a flashlight. One child thought that all he had to do was to touch the wire to the battery and to the bulb. One child patiently held the bulb against the battery. When asked what she was doing, Ellen replied, "Well, it's like a car, it's got to warm up before it works."

After their initial exploration, this project fostered a great deal of interaction among the children as they became curious about what their peers were doing; the result was a rich sharing. Their struggles were interesting to watch, as was the excitement that was generated when the first child discovered a way to make the bulb light up. Some children immediately asked the successful child how she had done it; other children ignored her and persisted on finding their own solutions. Yet other children insistently asked the teacher or one of the aides to show them.

Once the children, with or without assistance, figured out how to light the bulb, they began to generate their own experiments and projects. For instance, if one battery lights up a bulb to a certain degree of brightness, how much brighter would the bulb become with two batteries? Or three? Or four? The intrepid children who tried four batteries got a surprise: The light bulb popped. Oops! Some children quickly went on to other projects such as seeing if they could combine their resources and hook up more than one light bulb at a time. Other children continued to explore ways to make the light bulb light up. For instance, does the wire have to touch the knob on top of the battery, or could it touch anywhere on the top or the bottom? What happens if the wire touches the side? Does the base of the bulb have to be touched by the wire, or could any part of the bulb be touched? And some children simply sat there with their one bulb glowing, absorbed in the fascination of this phenomenon.

Observing their explorations led the teachers and aides to pose some basic questions: "How come we have to touch the battery and bulb in those ways in order to make it work? How come it doesn't work if the battery touches the coated part of the wire? What's inside the bulb that helps to make it light up?" How does the battery make the bulb light up? What do you think is inside the battery? What role does the little wire play?" In a sense, these questions seemed to give voice to the children's unspoken questions as they explored the materials and generated tasks for themselves. Some children attempted to verbalize their thoughts about causality in relation to electricity; others chose not to.

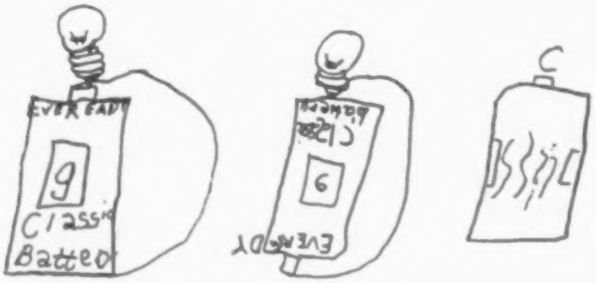
However, all children seemed to be taking in the spirit of the questions and reveling in the atmosphere of inquiry and exploration.

The next time the children came we again had the batteries, bulbs, and wires out, this time adding bulb holders as well. Upon arrival, the children came running over to the table where we had the materials set up, and once again began their explorations. Except for Ricky, who arrived with his dad. His dad proudly carried in a board with circuits and bulbs that he had made for his son. While Ricky was proud of what his dad had made, his own interest in the phenomenon of electricity had diminished. In this case, the father was nurturing his own talents instead of fostering his son's.

The first order of business for most of the children was to verify for themselves that they remembered how to make the bulbs light up. For some children, this involved the same series of mistakes as the last time (e.g., not putting the wire on both ends of the battery) and a reconceptualizing of the issue of what makes the bulb light up. Other children quickly got their bulbs to shine; they wanted to go on to other projects. For these children, we suggested that they figure out how to make more than one bulb light up with the same intensity using only one battery. This task appealed to them and they worked together to solve the problem. Three children managed to connect more than one light bulb. However, they immediately noticed that the two bulbs that lit up were dimmer than when just one bulb was lit and questioned why that would be. The dominant theory that they expressed was that the bulbs were sharing the electricity generated by the battery. They decided that what they should do was to add another battery. This led to some engineering exploration: How to make the batteries stay together without holding on to them. A quick search procured rubber bands which, with some finessing, did the trick.

Another group of children seemed to have decided that lights are to be used, not just explored. This group set about making a little pattern block village which they then lit up with their batteries and bulbs. Two other children created origami structures and inserted the lit bulbs inside their paper creations.

All the children were asked to draw a representation of their work with the batteries and bulbs. They were subsequently asked, "Which book (number, shape, or logic) are you going to put your drawing in?" Some children thought maybe the shape book would be best given the fact that batteries and bulbs possess shape. Other children were adamant that it should go in the logic book because, "You hafta do it in a logical way or the bulb won't light up."



Math Trek was drawing to a close for the summer, so we took up further elaboration of batteries and bulbs the following year. In one class, before any materials were set out, the teacher asked the children to draw their ideas about how the bulb is made to light up and then test their ideas by building their representations. Not surprisingly, there were many misconceptions. The children were asked to make a drawing that would correct the problem. This process of drawing their incipient models of how electricity works was quite productive. Pretty soon the children were discussing where the problems might be and articulating their theories to themselves and to each other. It took a different number of drawings per child, but eventually the drawings led to the satisfactory conclusion of lit bulbs.

In two classes, the exploration moved on to the question, "Which materials conduct electricity?" The teachers brought a variety of metallic and non-metallic materials and the children set about testing their theories about the conductivity of these different materials. One theory that came up repeatedly was the idea that materials that are silver conduct electricity. After all, wires are mostly silver, and the aluminum foil worked, too. The children were invited to test that hypothesis by using mylar. Before actually constructing the circuit, they were asked to make predictions. The children were fairly confident that the mylar would work. They were quite disconcerted when the light bulb did not light up; back to the drawing board, as it were. Renee ventured this: "I think maybe it isn't the color that matters. 'Cause the reddish wires (copper) work, too, and they're not silver. And boy, this stuff (the mylar) sure does nothing."

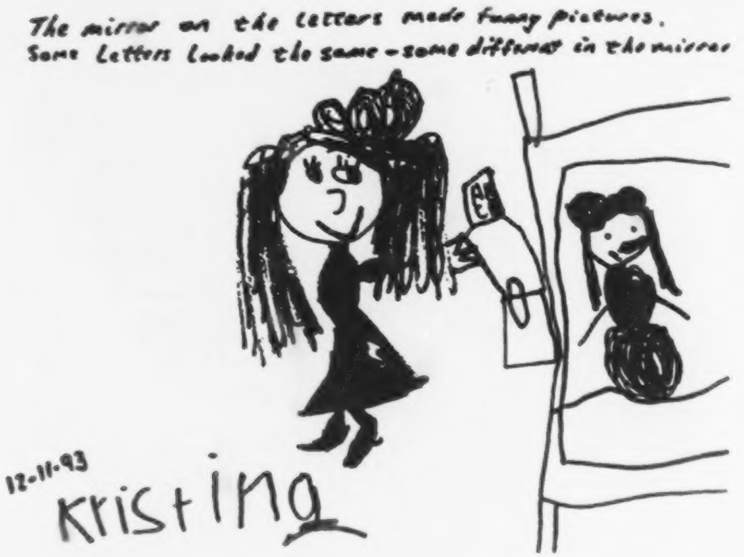
An aide interjected at this point, "So, do you have a rule then for what works and what doesn't?" Mark quietly said, "Maybe it's what the stuff is made of, not what color it is." Although the children were asked to expound more on that thought, they had enough for one session and were content to leave it at that. By the next class, however, it was clear that the children had continued to think about the puzzle at home. With or without parental help, they had grasped the essential idea that metal conducts.

After these two sessions, we moved on to other topics and activities. However, batteries and bulbs was a popular request for the activity part of the last session deemed "Children's Choice."

Math and Writing

Mathematics and writing have also become a dynamic duo in mathematics education. This pairing gives children the opportunity to communicate their reasoning and to struggle with the complexities of representing thought on paper. This struggle allows children to not only communicate to others, but also to themselves; in this way, implicit and intuitive notions about mathematics become explicit and better known.

At Saturday Club sessions we encouraged writing in several ways. At the end of each session we asked the children to write about something they had learned or particularly enjoyed that day. Because some of the children who attended were quite young (age five at the beginning of Saturday Club), we also encouraged them to draw a picture about what they learned if writing was not their preferred medium of expression yet. Drawing is actually an important entry point into writing (see Vygotsky, 1962) and helps children to see that representing thinking is a rewarding and communicative process. For older children, drawing lends itself to visualizing problems, a process which aids in solving the problems.


The mirror on the letters made funny pictures.
Some letters looked the same - some different in the mirror
12-11-93
Kristina

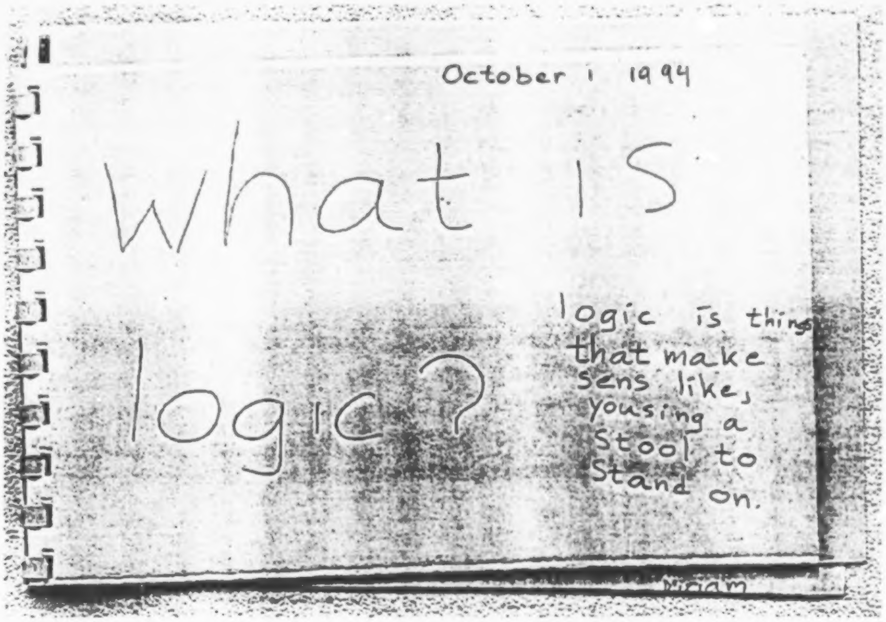
Another way in which we encouraged writing was to ask children to write stories for equations. Children are quite used to reading or hearing story problems for the various operations, but the request to write a story for an equation, to become the author of the story problem, was novel. The children were quite engaged by this task and typically both wrote stories and illustrated them. Over many weeks the children wrote stories for the following equations:

$$17 + 6,\ 19 - 11,\ 10 - 10,\ 0 - 3,\ 12 \times 24,\ 18/3$$

We purposely gave some equations that were no longer problematic for children (e.g., the addition and subtraction ones) in order to contrast stories and pictures used to represent prior understandings versus the use of stories and pictures to help create understandings. Creating a plausible story for zero take away three pushed the children to think through what negative numbers really mean in an actual life context.

Recording and Representation: A Problem to Solve and Another Way to Assess Understanding

While math journals are becoming a more common facet of math programs, we find that many children resist documenting their mathematical thinking, perhaps because they view writing as irrelevant to math, or simply because it is difficult to represent. At Math Trek, we worked hard to help participants find writing about math useful and interesting. We capitalized on children's natural affinity for ritual and celebration by framing the task of recording mathematical thinking as a ritual that we could celebrate each session. The celebration began when we asked children to make three books in which to record different aspects of their mathematical work: "My Book of Numbers," "My Book of Shapes," and "My Book of Logic" and to first consider why they might be making these books, and how they might be useful. Some of the responses were: "So we can put our favorite numbers and shapes in them!" "So we can talk about how to get certain numbers and how to make shapes." The participants generated ideas for other books, too, such as "My Book of Patterns" or "My Book of Number Facts." Intense discussion ensued concerning how there could be both number patterns and shape patterns, so there didn't have to be a separate book about patterns, and number facts could go into the book of numbers. Many children were puzzled as to what logic was and what might go in such a book. This puzzlement about logic led to a series of talks and problems about what constitutes logic and its opposite, including homework in which children had to write about an illogical situation (which became their first contribution to the "Book of Logic"). These discussions were a prelude to the sorts of discussions we had as to which book would be appropriate for a given activity, especially activities that combined numbers, shape, and logic. Even the choice of which book to use (and thus how to categorize the mathematics involved) became a problem-solving exercise.



October 1 1994

what is logic?

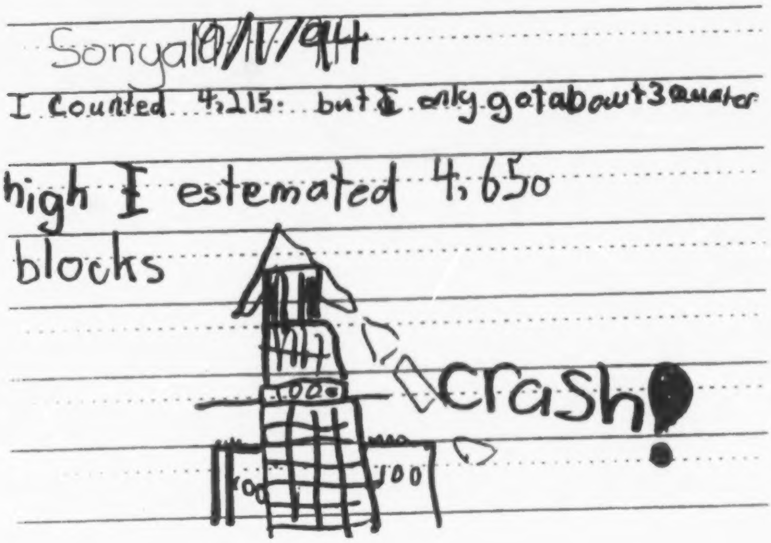
logic is things that make sens like, yousing a stool to stand on.

Noam

The books themselves varied with the teachers' ingenuity. They consisted of various materials such as large file cards hole-punched and bound with yarn, or paper stapled to heavy stock paper for cover. We asked the children to decorate these books with appropriate drawings that would convey what the books were about, and they responded with enthusiasm, designing the covers with great care and detail. The act of designing covers signaled that the books were special. These books were kept by the teachers and put out at the start of each session along with pencils, pens, and markers, for the children to use at any point. Graph paper, dot paper, and glue were also put out in case they were needed.

We made it clear that the books had several purposes: To express mathematical thoughts, to help solve problems, and to illustrate problem-solving processes. We also shared with the children the fact that their writing and drawing helped us to understand their thinking, and that they were our partners in research. Writing, of course, is an important tool for research and therefore gave children a *reason* to write. Having an audience in mind also motivates writing. Therefore, we suggested that they write for the children in the session that they did not attend. Especially because many children had only beginning writing skills, parents were called upon at the end of the session to help record the children's words.

Once or twice a session, at the end of an activity or at the end of the session, we requested that the children use their books to record and reflect on their activities. We varied the instructions, however, to keep the act of recording interesting, challenging, and novel. We also frequently individually tailored the assignments. For instance, after playing the Chip-Trading Game for the first time, we asked the children to make a drawing about the game. Lainie represented the game by drawing a picture of the playing mat, placing a filled-in red circle under the red column, and writing in, "I won!" After the children had more experience playing the game, we asked them to write or draw what the red is worth in the Land of Threes. This request stymied the children until they were encouraged to start drawing *as a way of solving the problem.* In this instance, the children were learning first-hand that representing is also a means of problem-solving.



Sonya 10/11/94

I counted 4,215 but I only got about 3 quater

high I estemated 4,650

blocks

We also had a ready stock of questions to help children who did not quite how to start writing. These questions included: "What did you learn in math today?" "What did you like best in math this week?" "What was hard for you?" "How could you make what was easy, harder?" "How could you make what was hard, easier?" "What would you like to learn about in math?" "Record how you solved the problem." "What's another way that you could have solved the problem?" "What did you learn from another child?" "Give instructions for the math game (pattern, etc.) that you made today, so that another child could follow those instructions."

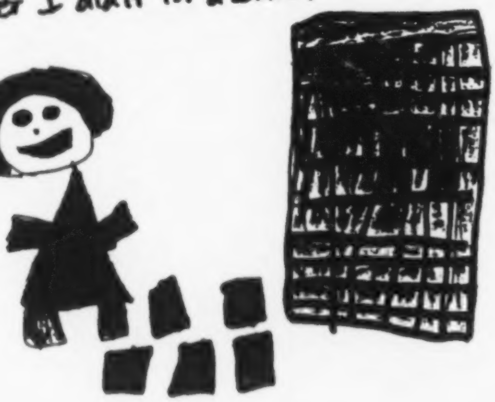
At times, children diligently worked with the math manipulatives, building tall towers, constructing intricate patterns, and fashioning shapes and animals. The journals became a method for challenging children to take their construction expertise and to think more deeply about it by drawing their creations in their books. Andy's base-10 block tower repeated a pattern of different-sized blocks that reached as high as the teacher. The teacher commented to him, "It's such a shame that we have to take that down at the end of the session. Could you make a drawing of it so you could make it again next week or perhaps so one of your friends could make the exact same one?" Andy shrugged, took a piece of graph paper, and worked for 20 minutes to faithfully represent his building. He captured the numeric value of the blocks by using the grids on the graph paper. He was proud of the results, and we gained insight into his spatial-visual abilities.

In another example involving a structure made from base-10 blocks, Ricky was initially resistant to the request to draw a picture of the structure. So Paul, the teacher, took a piece of graph paper and asked if he might draw the picture himself. Ricky shrugged, but agreed. Paul instructed Ricky to, "Make sure I'm doing it right" and commented, "From where I'm sitting, this is what I think your building looks like." In this way, the teacher subtly drew Ricky's attention to the issue of perspective. Ricky became intrigued enough to draw the structure himself. He made a series of five pictures in which he drew the building from each of the four faces as well as from the top looking down. His pictures were not to scale, but they did articulate spatial configuration and perspective.

It is interesting to speculate as to why some children do not want to draw or write. In this case, perhaps Ricky was stymied by the fact that his building was comprised of stacks with holes in them and uncertainty as to how to represent the holes. This was a child who was used to formulating answers speedily; perhaps his high performance mode was getting in the way of his taking a risk and doing something he did not already know how to do. The teacher's gentle manner of sitting down, drawing the building himself, talking about what he was doing, and inviting Ricky into the process, all helped to defuse anxiety and melt resistance to the task.

We were delighted to note that gradually the children no longer required as many prompts to use their books; often, they spontaneously grabbed the books during or after an activity because it became important to them to record, and a part of their mathematical work.

First it was fun. I made 9 shapes. Then it
started getting boring. Boring means you get tired of
doing it. After I didit for a while.



Assessment

The Math Trek teachers did not have any formal obligation to assess the children's mathematical thinking and behavior. However, in order to create and revise our curriculum, be responsive to the children, and provide occasional feedback to parents, ongoing informal assessment became an important feature of our work. Assessment was woven into each activity as we requested that the children articulate their reasoning, document their thinking processes in their journals, and engage in joint problem solving, and as we observed their engagement with materials, choice of job cards, and participation in group discussions. Their problem posing and extending also provided excellent grist for the assessment mill.

Our own record keeping after each session enabled us to document children's thinking and activities. Our meetings provided a forum for sharing notes and observations and for speculating about what particular mathematical behaviors actually mean. The different perspectives helped us to make sense of what we observed and heard. In many ways, this was an ideal situation, but one that could be replicated on a smaller scale in many schools and in very diverse settings.

CHAPTER 8: Character Profiles

Even in Math Trek there was a surprising variety of mathematical interests and talents. Among these five- to seven-year-olds, there were children who were computational wizards and loved to calculate; there were children who did not know any math facts but who were quite engaged by stories and drawings that related to mathematics; there were children who mainly loved construction toys, computers, and projects and hated to verbalize their mathematical thinking; and there were children who were game for anything. Thus, it quickly became clear that, as with any class, the teachers would need to adapt lessons, questions, and assignments to different personalities and cognitive styles.

What follows are short composite character sketches of some of the different types of children who came to Math Trek. This is by no means an exhaustive study of mathematically talented children; it is simply an attempt to express various flavors of talent that you may well recognize from your own classes. The other purpose of these sketches is to provide some strategies for meeting children on their own terms and extending more personal invitations to come along on mathematical investigations.

Creative and Artistic

We'll start with a shy, quiet kindergartner who preferred listening to talking. It was clear from early on that she loved to draw, loved stories, and loved a good game. Her mom noted that after she had learned to play the Chip-Trading Game, she made the game at home and taught her parents how to play, a clue for us that if her interest was piqued, she would take the initiative to play with and extend the relevant ideas.

A magical turning point occurred during our study and discussion of numerals. As part of that study, during the middle of the first year, the children were asked to invent a numeral character. Carolyn's character was Zif, the name referring to the letter Z, for at that time Carolyn thought that perhaps numerals and letters were the same thing, that is, interesting visual signs that are very important to teachers and adults, but a little mystifying to five-year-olds. Carolyn's creativity was unleashed by the opportunity to create a character. Over a period of 8 months, spanning from one Math Trek year to the next, she created, on her own, Zif's entire extended family and their several homes as well as stories about how they related to each other. She brought in her latest creation each time she came to Math Trek, using the picture or creation as a way to reconnect to her teacher and to the group. During this time, she sorted out the difference between letters and numerals and explored all the ways in which number can be used to describe families, and shape can be used to create homes and the objects in those homes.

Knowing that stories and drawings were important to Carolyn, her teacher often suggested projects to help her get going when she first arrived at Saturday Club. For instance, one Saturday her teacher asked Carolyn for her favorite number. She promptly responded, "17!" After a short discussion of why 17 was her favorite number, her teacher suggested that she make a book about all the ways to get to the number 17. She was delighted with the suggestion and promptly set to work. For the next 40 minutes she created a book exploring many equations that yield the answer, 17. She also decorated the front of the book with an elaborate "17," once again showing her proclivity for drawing and rendering even objective numerals personally meaningful.

Another time, pattern blocks were set out as one choice for the initial activity time. Carolyn quickly made an animal out of the pattern blocks. She loved to show her teacher

her creations and would wait patiently for her attention if it was engaged elsewhere. After she showed her animal her teacher suggested that she figure out a way to show her pattern-block animal using rubber bands and the *Geoboard*. Her eyes grew wide at this new possibility and she set to work again, this time trying to figure out how to represent something three-dimensional in a two-dimensional form.

Carolyn happened to be in a group that had many children a year older. In order to meet all the needs in the class, there were times when the material seemed to go over Carolyn's head. For instance, when we read *The Mysterious Multiplying Jar*, she soon moved away from the circle of children and went off to a nearby desk to work on a drawing project, although she seemed to be listening all the while. When asked about it later she said, "That book made me dizzy. I think I'll read it later." This was a nice example of self-regulation in a learning situation. She took in just as much as she wanted and felt free enough to pursue something more relevant to her interests at the time.

Toward the latter third of the second year, Carolyn had become a much more vocal participant. During group problem-solving sessions, she piped up with her solutions and understandings. She loved to share her projects during the sharing that occurred after the initial activity time each session. She often paired up with Sarah, a child in the next grade. This was a good combination; Carolyn was stimulated by the more sophisticated activities of the older child, and Sarah, an only child, was very pleased at the attention and the opportunity to be a leader.

Computational Wizardry

Craig was a remarkable little fellow, who started Math Trek as a kindergartner. The first time his teacher met him he proudly announced, "I'm working on a 7th grade math book!" His teacher asked, "Oh. What are you enjoying learning about?" He said, "Algebra!" His teacher was intrigued: "What is algebra?" At that point, with characteristic five-year-oldness, he ran off to play with the *Googleplex*—a construction toy comprised of pentagons, wheels, and other shapes.

His favorite activity with the *Googleplex* involved making wheels and then rolling them in the hall. He was particularly interested in making a complex combination of wheels attached by a *Googleplex* gizmo sturdy enough that it wouldn't fall apart on impact as it was rolled vigorously down the hall. Although his preference was to work by himself, his *Googleplex* constructions and stability testing served to bring other children into his orbit.

Interestingly, Craig was willing to talk about his construction activities, and could articulate goals and theories about how to reach that goal. He just didn't want to engage in such talk about computations or numerical kinds of problem solving. One of our goals for Craig was to help him wed his considerable interest in numbers and the manipulating of them to more contextualized and verbalized problem solving.

There were two remarkable incidents with Craig that heightened the importance of the goals we had set for him. One involved solving a multiplication word problem in this context: There are 12 Christmas trees with 64 lights on each tree; how many lights in all? Craig preferred to work by himself. When his teacher came over to him, about ten minutes after the problem solving had begun, he was saying aloud, "Well, let's see, I think I know what 12 x 4 is. Hmmm. Let me just think about it some more, 12 x 4, 12 x 4." His teacher asked, "Do you think you could figure out what 12 x 4 is? Instead of trying to remember what it is?" Craig shook his head. It was then suggested that he pair up with another child, Peter, about the same age, who had figured out his own strategy for getting the answer. Craig briefly glanced at Peter's work and gave up on the problem.

Reflecting back on this episode, it seemed apparent that Craig was not yet ready to learn from or to teach peers. We wondered if he would engage more if he worked with an adult who could tune in to him and provide an appropriate level of stimulation. One of the male college students who came each session began to spend a little more time with Craig, especially during the initial activity. Craig thrived on this attention, especially since it came from a young adult male. Together they explored issues such as the area of squares and triangles and how they relate, batteries and bulbs, and golden rectangles. Craig gradually became more willing to apply his computational knowledge to word problems and other contexts.

Toward the end of the second year, the following problem was posed on the board for one of the several choices of initial activities: "Find a number that's less than 70 and whose factors are 3, 4, 5 and 6" (adapted from Schifter & Fosnot, 1993). Most of the children chose a different activity or problem. One girl, Tanya, was still having some difficulty separating from her mother, so her teacher used the problem to help Tanya separate. This was always a tricky matter; Tanya had chronic severe separation anxiety and her mother often became so engaged in the activities and problems that she, too, didn't want to leave. She also had a tendency to help Tanya figure out a problem by telling her what strategies to use. Here, too, Tanya's mom became quickly engrossed. She began by asking Tanya if she knew the trick to figure out whether any given number had a factor of 3. The teacher gently reminded the Mom that Tanya could probably figure out her own way to solve the problem and would be glad to hear the mother's trick *after* she had had a chance to figure it out on her own. At that point, the Mom went off to read in the hall.

As soon as Craig came in, the teacher asked his help in solving the problem. The two children barely took notice of each other as they worked on the problem. In an attempt to facilitate joint problem solving, the teacher spent some time calling Craig's attention to what Tanya had just said. Craig did not seem to be listening to her at all until she exclaimed, "Oh, the number would have to end in a 0 or a 5 if 5 is going to be a factor of that number." Craig looked up at her curiously and said, "Yeah, because if you multiply 5 it always has a 0 or a 10 in the answer . . . then that would mean we could eliminate all the other numbers. Let's figure out if 65 is the answer." Pretty soon, they were collaborating, feeding off each other's ideas, and listening to each other's strategies. This joint problem-solving session ended successfully—the problem was solved, as was the issue of collaboration.

It is interesting to note that this factor problem involved only number relationships; it did not refer to any real life context or require a known algorithm. Pulling away from algorithmic knowledge and not demanding applied problem solving seemed to offer Craig just what he needed, both in the realm of problem solving and collaboration. Sometimes the answer lies in finding just the right problem.

Marching to a Different Drummer

Danny was a lively, articulate, self-directed youngster whose intuitive knowledge of the number system was as advanced as that of any child in Math Trek. His behavior was, however, taxing to say the least, until we learned to adapt. He was initially placed in an afternoon group because of his age, but he always arrived in his father's arms, sleepy or sleeping, and loudly voicing his objections—until we supported his choice to go home, at which point he insisted on staying. (Only much later did we learn that he was taking antihistamines for allergies, which no doubt contributed to the sleepiness.) We invited the parents to bring him in the morning, but the afternoon schedule fit better into their plans. Danny's family was experiencing a good deal of tension and this stress may well have contributed to his need to test limits, to be best at everything, to meet life on his own terms.

Danny was always grumpy and unwilling to participate at first, so we let him "nest" under a table until he was more alert. Even after he was thoroughly awake, he remained rather grouchy and hung back, actively rejecting our invitations and seldom making choices or even engaging in activities he might devise for himself. We guessed that Danny was rather used to being told what to do at home and had not had much "mess around time" to explore interests of his own. Half-watching ongoing games and projects from the sidelines, he fiddled about the room rather aimlessly until we were ready to put everything away to go on to the next activity. That was the point at which Danny decided to do the first one. Clearly, we realized, he needed his own plan. Eventually, we learned to place some interesting books under the table before he arrived and to leave him to his own devices. We spent no energy cajoling him to join us and simply let him know what was coming next.

This system worked beautifully for Danny. Having had quite a lot of time to get himself ready, he generally joined us for the second group activity of the day and, once engaged, he grasped concepts quickly, often became intensely invested, and was excited and enthusiastic about his new discoveries. Because Danny was never ready for recess when the others were, one of us stayed inside with him as he continued to explore materials that had caught his interest and to write or dictate detailed notes for his journals. Danny tended to work by himself rather than with the other children, anyway, for it pleased him very little if they arrived at a solution or finished a project before he did.

That Danny's tempo resulted in his skipping many opportunities was a tradeoff to which he, and we, needed time to adjust. But from "Difficult Danny" he became "Different Danny," marching with verve and precision to every other beat of his own drummer!

Anything But Writing!

JoAnne hated writing. The worst parts of first grade for her were all the requests to write. Her mom was puzzled by JoAnne's dislike of writing, for she loved to read and draw. Her favorite subject, however, was math. During one of the second year Math Trek sessions, the children were asked to make a drawing and write a story that would make sense of some simple equations. One equation was 0 - 3 = -3. JoAnne loved negative numbers and was intrigued by the challenge of coming up with a plausible story. She spent a long time drawing a picture and then wrote a comical story about a man who had to dig three levels underground in order to get to a certain pipe.

After this episode, JoAnne would often choose to write a story about an equation instead of other activities offered. Connecting writing to a subject she cared about gave her a reason to write. While other children may not share JoAnne's initial feelings about writing, writing about math is an excellent way for children to make sense of equations and to invite them to reflect on math.

Diligent, Hard-Working, and Perfectionist

Pam was one of the hardest-working, most deliberate, and thoughtful participants at Math Trek. She came into the sessions ready to work at whatever was put before her, erasing and starting over if the work was not neat and precise enough for her. Typically, she liked anything to do with calculations or patterns. She hated all of the estimating activities and refused to volunteer an estimate. In group discussions, she never volunteered her thinking and raised her hand only when she was sure of an answer.

One session, she worked on number palindromes derived by adding a pair of palindromes, and then adding sums to addends until the next palindrome is accomplished. She filled the entire board with her calculations. When her dad came in to pick her up, we showed him Pam's prodigious efforts. He glanced at the board and then pointed to one calculation in the hundred thousands and said, "There's a mistake there." It was no wonder that Pam tended not to be a risk-taker!

Although it's hard to know how ingrained some personality characteristics such as perfectionism are, we thought it would be worthwhile to help Pam take some intellectual risks. We were careful not to pressure her and to back off when she resisted. Our tack was to make sure she had some one-to-one time at some point during the session, often at recess time. Pam was an excellent gymnast and could perform amazing backward flips and splits. After admiring her accomplishments, we would casually ask her to estimate how many flips she could do at once or how many somersaults it would take her to get across the gym floor. Eventually, we invited her to take on the all-important role of "mother" in the game of "Mathematical Mother May I." For this role she was required to invent math problems to give to the other children. Gradually, we noticed that Pam began to take more risks, even volunteering opinions and answers during joint problem-solving sessions as when we read Open Court stories.

High-Spirited and High Energy

Kenny was as impulsive as Pam was deliberate and careful. He came into sessions like a ball of fire, needing to touch all the materials, read all the job-cards, and make contact with all the people in the room. When asked to estimate or solve a problem he was always first to raise his hand (and was often right!), and during group discussions was often the first child to become squirmy and complain that he was bored. Keeping Kenny engaged and invested was challenging, to say the least. He was obviously very bright as well as very restless.

Observing him made it clear that he needed a lot of space and leeway which we could offer him as long as he was being polite and appropriate. Pairing him up with one of the aides was also very helpful, as he enjoyed having an audience for his mathematical prowess. Activities such as the batteries and bulbs project were very successful in sustaining his attention because of the multiple elaborations with which to experiment. We also discovered that he was quite talented at representing his mathematical thinking and activities through drawing which we encouraged him to do each session. He was definitely our signal system; if Kenny was bored we knew an activity was not working out quite right and if he was thoughtfully engaged we knew we had a hit!

Epilogue: A Reminder

This book has been written with the goal of making mathematics stimulating and rewarding for math-talented children. In Chapter 3, we discussed ways of organizing classes and schools to support their development; Chapters 4 through 7 described a philosophy of teaching mathematics and open-ended teaching methods that make it possible, within regular or specialized classrooms, to engage and challenge children to learn about mathematics in ways that are deep and resonant. These open-ended activities allow children to seek their own level, affording many possibilities for the expression of talent and interest.

We want readers to be well aware that it is not the specific tasks we have described that constitute the meat of the matter, for they are only examples and illustrations that imaginative and spirited teachers can use to spark their own ideas. There are a great many other sources of teaching activities available these days (some embedded in current curricular materials, some examples listed in the Annotated Bibliography in the appendices to this volume; and many others published by the National Council of Teachers of Mathematics and other publishers and journals such as *Teaching Children Mathematics*).

Rather, the message of this book is that all children learn best when the challenges provided in school are those for which they are just about ready. Accomplishing this *optimal match* follows naturally when teachers are open to seeing children as individuals, ask intriguing questions to help children pose problems, supply information when children are ready (and hungry for it), and listen carefully to children's thinking. In these ways, teachers can maintain the flexibility that permits the math-talented child to be challenged and empowered and for all the children in the class to have exciting math experiences. For, of course, some mathematical voices are stronger and richer than others; it is the teacher's responsibility to pull all the voices together into a community of learners where each child gets to hear and learn from others. In this way, teachers listen to and respect individual voices in the community.

It is our hope that, having read this book, teachers will make use of the materials included in the appendices in the service of their own skilled approach, in the context of a "gifted-friendly" classroom—a classroom that is equally friendly and responsive to all children.

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Appendix A

Questionnaire for Parents

Questionnaire for Parents

Date ___________________________
Completed by Mom ____ Dad ___

MATH TREK PARENT QUESTIONNAIRE

1. How old was your child when you first noticed a special interest in numbers or number ideas, more than you expected at his/her age?
   ___ Yrs ___ Months; (or) ___ I haven't noticed any such interest

2. If you have noticed such interest, please describe what you remember about your child's first involvement with numbers.

   Subsequent incidents you remember?

3. Did you try to teach your child about numbers before he or she showed a spontaneous interest? ___ Yes ___ No. If so, what?

   After your child showed some interest, and you began to respond to that, do you remember what you did to encourage your child's interest? Please describe. (For example, did you pick out books that featured counting? Talk about numbers? Play board games?)

4. Is anyone in your family known as especially "good at numbers?" Please describe.

5. Does either parent have a job that involves working with numbers? Please describe.

6. Now, about your child's current skills with numbers. Do you think your child could typically do these things? (You need not ask the child actually to do these before you answer):

Yes No ?
___ ___ ___ Repeat his or her street address?
___ ___ ___ Phone number?
___ ___ ___ Remember anyone else's phone number or address?
___ ___ ___ Count 20 things (not just saying the words)?
___ ___ ___ Spontaneously comment on number relationships, like, "A bird just flew away; now there are only 3!"
___ ___ ___ Spontaneously comment on signs such as speed limits?
___ ___ ___ Tell how fast your car is going by the speedometer?
___ ___ ___ Play a board game with counting (e.g., Parcheesi)?
___ ___ ___ Play a complicated game like Monopoly (no help)?
___ ___ ___ Add two-numbers up to 10?
___ ___ ___ Add two-numbers up to 20?
___ ___ ___ Add two-digit numbers without carrying?
___ ___ ___ Figure the difference between 9 and 2?
___ ___ ___ Figure the difference between 9 and 21?
___ ___ ___ Count by 10s to 100?
___ ___ ___ Count by 100s to 1000?
___ ___ ___ Make change for a quarter?
___ ___ ___ Make change for a dollar?
___ ___ ___ Follow a recipe calling for measurement (no help)?
___ ___ ___ Tell whether a nickel or a dime is more money?
___ ___ ___ Tell the days of the week?
___ ___ ___ Tell which day comes before Friday?
___ ___ ___ Know the meaning of "last week" and "next week?"
___ ___ ___ Do problems in math workbooks? Grade level ____
___ ___ ___ Tell which is smaller, 6 or 4?
___ ___ ___ Tell which is bigger, 33 or 27?

7. Is there anything else you'd like to mention? Please feel free to continue on the back of the page if there is something we have missed!

Appendix B

Job Cards and Other Activities

Job Cards and Other Activities

Introduction. This Appendix gives some examples of the types of *Job Cards* and games that we used at the Saturday Clubs. We usually set out one type of card per table, along with an array of material that might be useful tools for solving the problems detailed on the cards. At times, we included extension activities and questions that helped prolong the math exploration.

What Is Your Name Worth?

If A = one penny, B = two pennies, etc., what is your first name worth?

What is your last name worth?

How did you figure it out?

What is the most expensive name you can think of?

Think of a name that costs exactly forty-three cents!

Can you think of any variations on this?

Write a Story for . . .

1. Write a story for this number sentence:

   0 - 3 = -3

2. Now draw a picture to go with your story!

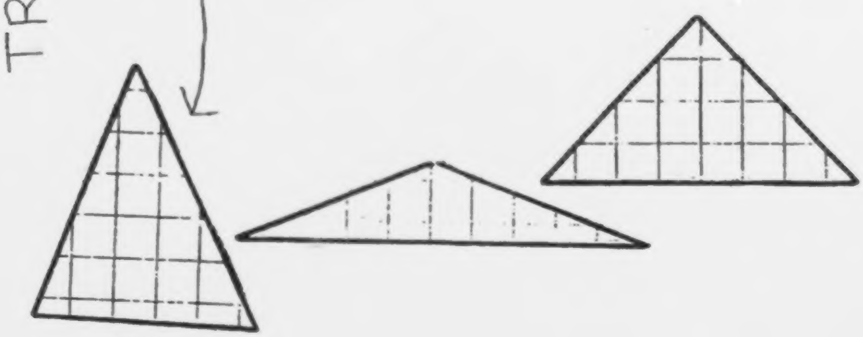
Triangles!

1. Draw 3 different kinds of triangles.
2. What makes them all triangles?
3. Which triangle is your favorite one? Do your classmates agree?

TRIANGLES

2. They all have 3 sides. 15/1/94

3. This is my favorel triangle becaus this might sound funny But it looks like a roof-top.

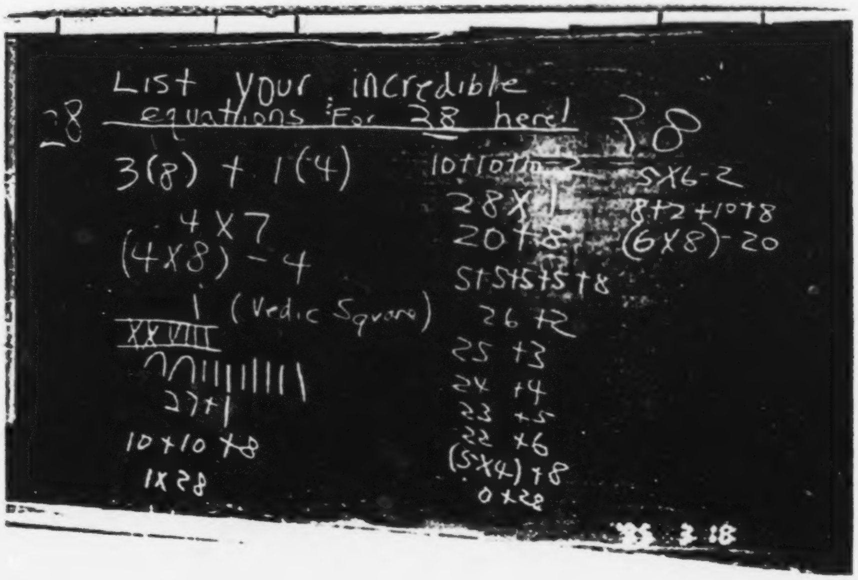


Incredible Equations

Take a number like 28 and see how many equations you can write that will have that answer.

Challenge: Making equations make sense.

Make up interesting equations that at first glance don't make sense because we need to think first about the units involved. For example, in what way is **5 + 5 = 2** true? How about **1 - 1 = 59**?


28
List your incredible equations For 28 here!
28
3(8) + 1(4)
4X7
(4X8) - 4
(Vedic Square)
XXVIII
27+1
10+10+8
1X28
10+10+10-2
28X1
20+8
5+5+5+5+8
26+2
25+3
24+4
23+5
22+6
(5X4)+8
0+28
5X6-2
8+2+10+8
(6X8)-20
'95 3 18

Pentominos

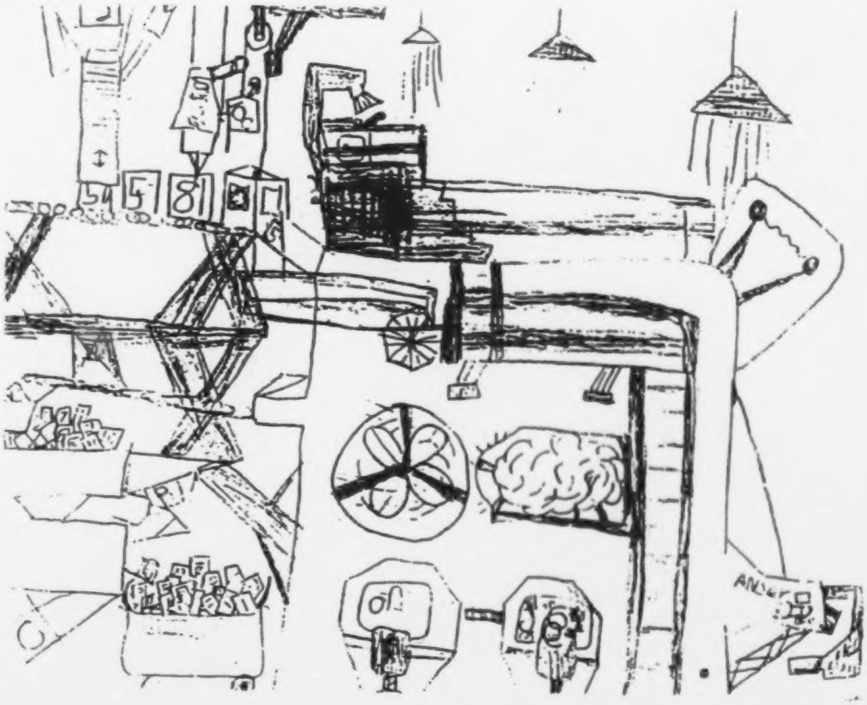
Take five tiles and arrange them in as many different ways as you can, making sure that at least two tiles share a side.

Rotations don't count as a different arrangement.

Draw each arrangement you make on a piece of graph paper.

How many arrangements did you make all together?

Name each arrangement!



Switching Places Boat Problem

First, draw three circles big enough to fit color tiles on to:

O O O

Take 1 blue and 1 yellow tile and place them on the two outside circles. Imagine that the paper is a boat with three places, and the blue person wants to switch places with the yellow person. They can only switch by moving one position at a time, and in one direction. They can also jump over one person, but only one of an opposite color.

After you have figured this out, draw 5 circles and use two blue and two yellow tiles, leaving the middle circle blank.

Record how many moves it takes to switch sides.

Try this game with 7, 9, and 11 circles.

Record the following information in a table: Places in the boat. How many moves it took to switch places. Do you notice any patterns?

What's Your Pattern Worth?

1. Make a pattern with your pattern blocks.
2. Figure out how much your pattern is worth:

Cost of Shapes:

triangles = $1
squares = $2
parallelogram = $3
rhombus = $1
trapezoid = $4
hexagon = $5

Challenge: Can you make a shape that is worth $23?

What is the most expensive hexagon that you can make?

Pattern Block Fractions

Hexagons = 1 whole
Triangles = 1/6
Parallelograms = 1/3
Trapezoids = 1/2

Using only the above blocks, make a design and figure out how much it's worth.

How did you add and keep track?

How Many Different Kinds of Rectangles Can You Make With 12 Cubes or Tiles?

1. Make as many different rectangles as you can and record each one.
2. Describe each recording by saying how many cubes long, and how many cubes wide.

Now try it with 24 cubes.

Now try it with 19 cubes.

(And any other number that you like)

What did you discover from doing this activity?

Rectangle Hunt

Go on a rectangle hunt around the room.

Record the number of rectangles that you find, and their measurements.

Or, use graph paper and draw the rectangles to scale (for example each inch = 1 box).

Which rectangle(s) do you like best?

Why?

Golden Rectangles

1. Using graph paper, make a 13 x 21 rectangle.
2. Make the largest square you can in the rectangle (using the left side of the rectangle as one of the sides).
3. Going counter-clockwise, continue to make the largest square you can in the leftover part of the rectangle.
4. What is the size of each square?
5. Can you find the pattern in this sequence?
6. Draw diagonals, starting at the bottom left hand corner, through each of the successively smaller squares, using a continuous line.

Math Balance

1. Using a math balance, find three ways to balance the number 35 (3 tens and a five).
2. What can you say about the four different ways to make 35?
3. Make up problems and solve them.

Hexagon Game

Materials: Triangles, trapezoids, parallelograms, and hexagons from Pattern Block Sets are used for this trading game that involves fractions and congruency. One die, numbered 1 through 6, or a die marked with fractions (1/2, 1/3, 2/3, 1/6, 5/6 and 1).

Object of the game: To make as many hexagons as possible during the length of the play.

Playing the game: Triangles are the unit for this game, thus, the number rolled tells how many triangles or their equivalent to take. For example, if a three is thrown, the player can choose three triangles, a parallelogram and a triangle, or a trapezoid. On the next turn, a four is thrown. Now, the player can choose four triangles or two parallelograms or a trapezoid and a triangle. He/she can then figure out how to form a hexagon using the pieces he/she has. He/she will have a hexagon and one triangle left over.

Note: Prior experience with making hexagons from different combinations of pattern blocks is necessary.

This game works well done cooperatively. The group challenge is to fill a honeycomb pattern made by tracing hexagons on a piece of paper.

Measuring Angles with Pattern Blocks

Prior experience: Unstructured "play time" with pattern blocks, exploring how they fit together.

1. Start off by asking: "Why do pattern blocks fit together so well? List their ideas on the board, using their language.
2. Drawing on what the children say, introduce the idea of angle and draw an angle, mentioning how angles encompass both corners and sides.
3. Show them the various pattern block pieces and point out some of the angles. Ask: "What do you notice about the different angles?" List their ideas on the board.
4. Ask how they might categorize the different angles that they see.
5. Tell them that there is a way to measure angles; draw an angle on the board and show the area that is measured.
6. Hold up a square piece and ask them what they think about that angle. Tell them the angles on a square are right angles and measure 90 degrees. Ask them where they see right angles and why they are so prevalent (this should lead to a discussion of art, architecture, and nature). You might use their discussion of right angles to categorize angles as smaller than or larger than right angles (acute or obtuse).
7. Now, ask them to find out what all the angles on the pattern blocks measure (either working in pairs or in groups) using the information that a right angle measures 90 degrees. Tell them they must record each shape (have templates available) and show what each angle measures, and how they figured it out. (Give out a set of pattern blocks to each pair or group.) Before they start, ask: "Are all the angles in a shape the same?" Discuss the fact that they aren't all the same, so sometimes they'll need to figure out more than one angle for a shape.
8. If some children finish earlier, tell them to go on to figure out how many degrees are in the entire shape (by adding the degrees from each angle in the shape).
9. Discuss findings/strategies with the whole group. Have them show on the overhead with overhead pattern blocks.
10. Make a chart with the number of angles and the number of total degrees.
11. Fill in some of the holes on the chart by having them construct and figure out the degrees in a pentagon, a septagon, an octagon, etc.
12. When all the numbers are up there and discussed, have them find the pattern.

Lots of Boxes

Object of the game: This is a game for two people. The idea is to make a bigger rectangle than your partner.

Directions:

1. Each partner takes a piece of grid paper and a pencil.
2. One person takes the die and throws it. The number on the die tells you how *long* your rectangle will be. Now draw it. Then you throw the die again, and that will tell you how *high* your rectangle will be. Now draw it, and finish your rectangle. How many little boxes are in your rectangle? That's your score. Write it down.
3. Now it's your partner's turn to do the same.
4. Whoever has the bigger score, wins.

Play as many times as you like!

Questions to ask:

1. How did you figure out how many little boxes were in your rectangle?
2. Could you find an easier way to figure it out?
3. Could you write a number sentence to show how many little boxes there are altogether?
4. What's the smallest rectangle you could make by throwing the die twice?
5. What's the biggest rectangle you could make?

Challenges:

1. Use two dice each time you throw!
2. Devise two ways to figure out how many squares there are in your box.
3. Write number sentences or equations for each way.
4. Write a formula for finding out how many squares there are no matter what you roll.

Appendix C

Annotated Bibliography

Annotated Bibliography

Children's Literature

Anno, M. (1983). *The mysterious multiplying jar* (1-6). New York: Philomel Books.

Provides wonderful pictorial illustration of factorials.

Anno, M. (1982). *Anno's math games I.* New York: Philomel Books.
Anno, M. (1985). *Anno's math games II.* New York: Philomel Books.
Anno, M. (1987). *Anno's math games III.* New York: Philomel Books.

These books include a variety of intellectual games that invite children to see how mathematics permeates the world.

Anno, M. (1989). *Anno's hat tricks.* New York: Philomel Books.
Anno, M. (1990). *Socrates and the three little pigs.* New York: Philomel Books.

These two books explore probability.

Burns, M. (1994). *The greedy triangle* (K-3). New York: Scholastic.

The story of a triangle who magically changes shape in order to see what it's like to have more sides and angles. Good for discussion of how shapes are part of our world.

Draze, D. (1990). *Can you count in Greek: Exploring ancient number systems* (1-6). San Louis Obipso, CA: Dandy Lion Publications.

Explores Egyptian, Babylonian, Roman, Greek, and Mayan number systems.

Eichelberger, B., & Larson, C. (1993). *Constructions for children: Projects in design technology.* Palo Alto, CA: Dale Seymour Publications.

Includes instructions for projects from bridges to clocks with gears.

Schwartz, D., & Kellogg, S. (1989). *If you made a million.* New York: Scholastic.
Schwartz, D., & Kellogg, S. (1985). *How much is a million.* New York: Scholastic.

Shows how to play with astronomical numbers and what they look like.

Scieszka, J., & Smith, L. (1995). *The math curse.* New York: Viking Press.
Tompert, A., & Parker, R. A. (1990). *Grandfather Tang's story.* New York: Crown Publishing.

Contains story of animals transforming; based on tangrams.

Books on Children's Mathematical or Scientific Learning

Duckworth, E. (1996). *The having of wonderful ideas and other essays on teaching and learning* (2nd ed.). New York: Teachers College Press.

Duckworth has wonderful ideas about how children learn and how to understand children's learning and sense-making.

Gallas, K. (1995). *Talking their way into science: Hearing children's questions and theories, responding with curricula.* New York: Teachers College Press.

Includes reflections and examples by a first/second grade teacher.

Ginsburg, H. (1989). *Children's arithmetic: How they learn it and how you teach it.* Austin, TX: Pro-Ed.

This book explains how children's mathematical thinking develops, beginning in infancy. Great examples of how children invent meaningful ways of approaching numbers and calculations.

Kamii, C. (1984). *Young children reinvent arithmetic.* New York: Teachers College Press.

Kamii, C. (1989). *Young children continue to reinvent arithmetic: Second grade.* New York: Teachers College Press.

Kamii, C. (1993). *Young children continue to reinvent arithmetic: Third grade.* New York: Teachers College Press.

These three books describe in detail how children invent their own mathematically meaningful ways to compute with numbers. Based on the author's own classroom research.

Papert, S. (1993). *The children's machine.* New York: Basic Books.

Papert, one of the inventors of Logo, a computing program for children, writes about his vision for how children learn best.

Schifter, D., & Fosnot, C. (1993). *Reconstructing mathematics education.* New York: Teachers College Press.

Describes how several teachers changed to a more open-ended and constructivist approach towards teaching math.

Curriculum Materials

Burns, M. (1987). *Math solutions* (K-3). New Rochelle, NY: Math Solutions Publications.

Provides detailed descriptions of lessons that the author conducted in classrooms over several sessions each.

(Note: Marilyn Burns is a prolific author, with many worthwhile titles.)

Childcraft. (1988). *The how and why library* (Vol. 13: Mathemagic). Chicago: World Book.

This book explores the history and magic of mathematics through stories, puzzles, and activities.

Garland, T. H. (1987). *Fascinating Fibonacci: Mystery and magic in numbers.* Palo Alto, CA: Dale Seymour.

Explores the Fibonacci sequence in nature, anatomy, art, and architecture.

Haag, V., Kaufman, B., Martin, E., & Rising, G. (1995). *Challenge: A program for the mathematically talented* (3-6). Reading, MA: Addison-Wesley.

This is a program for teaching logic through puzzles and problems as well as other aspects of mathematical thinking and problem-solving.

Kaye, P. (1987). *Games for math.* New York: Pantheon Books

An inventive educator has come up with great ideas for how to play with math.

Magarian-Gold, J., & Mogenson, S. (1990). *Exploring with color tiles* (K-3). White Plains, NY: Cuisenaire.

Provides activities using tiles to explore operations, perimeter, and probability.

National Council of Teachers of Mathematics. (1990-1996). *Curriculum and evaluation standards addenda series.* Reston, VA: Author.

This is a wonderful series that deals with number sense, patterns, making sense of data, spatial sense, etc.

Ritchhart, R. (1995). *Making numbers make sense: A sourcebook for developing numeracy.* Reading, MA: Addison-Wesley.

This resource book uses interesting problem-solving contexts in which to investigate traditional topics in math such as place value, measurement, and statistics.

Stenmark, J., Thompson, V., & Cossey, R. (1986). *Family math* (K-8). Berkeley, CA: Lawrence Hall of Science.

Activities for all areas of mathematics that can be done at home with parents and siblings.

Winebrenner, S. (1992). *Teaching gifted kids in the regular classroom.* St. Paul, MN: Free Spirit.

Strategies and techniques to meet the academic needs of gifted children in regular classrooms.

NOTE: Dale Seymour and Cuisenaire put out catalogues that include many excellent titles. TERC, a research group, also publishes books that detail curriculum units to be used in conjunction with hands-on math and science learning.
TERC's address is: 2067 Massachusetts Avenue, Cambridge, MA 02140

Books to Satisfy Your Own Mathematical Curiosity

Peterson, I. (1990). *Islands of truth: A mathematical mystery cruise.* New York: W. H. Freeman.

Peterson, I. (1988). *The mathematical tourist: Snapshots of modern mathematics.* New York: W. H. Freeman.

Stewart, I. (1992). *Another fine math you've got me into.* New York: W. H. Freeman.

Appendix D

Recommendations for the Classroom

Recommendations for the Classroom

These recommendations summarize much of what we learned about how to work with young children and math. Some of the recommendations concern setting a climate that empowers, and other recommendations concern the curriculum.

1. **WAIT TIME:** "I'm interested in what everyone in this class is thinking. So I'm going to wait until everyone has thought through this problem."

2. **ALTERNATIVES WHILE WAITING:** Give the speedy children other problems to work on, or ask them to come up with more than one way to solve the problem.

3. **TALK TO CHILDREN ABOUT ALL THE DIFFERENT WAYS OUR MINDS WORK:** Some children have minds that work quickly and memories that allow them to recall facts right away; other children need more time but can often, given that time, come up with thoughtful responses. Have children observe themselves and each other and discuss the fact that minds work differently and that is okay. Ask about students' theories of what it means to be smart and introduce the idea that being smart involves working hard. Ask children to notice the wonderfully varied ways minds work as they share strategies for solving problems.

4. **EXPLORATION TIME:** Whenever you introduce a new material, or bring out a material children haven't used for a while, give children plenty of time to explore.

5. **OBSERVE AND EXPRESS INTEREST IN EXPLORATIONS:** Children are thrilled when you take an interest in their problem-posing and ask them questions and suggest extensions. You could also ask them to "teach" you how to do what they are doing, for instance, making a pattern with symmetry.

6. **USE THEIR EXPLORATIONS AS THE BASIS FOR CURRICULUM DEVELOPMENT AND DECISION-MAKING:** Sometimes new questions, problems, and activities can arise out of the children's explorations; children can teach each other, and you can plan further activities based on what you see.

7. **DEVELOP AND FOCUS ON THE BIG IDEAS:** Most of early mathematics revolves around key themes and big ideas. Study the curriculum in use and find the big ideas that underlie the tasks and activities. The focus on the big ideas helps children to sustain intellectual interest and excitement, and to make mathematical connections.

8. **ASK BIG QUESTIONS:** Children are often excited and intrigued by big, open-ended questions that don't have one definite answer, questions that have a philosophical bent. These are fundamental questions that invite children to theorize and become intellectually curious. Examples are: What is a pattern? Where do the patterns exist? In our heads or in the numbers? What is infinity? Can numbers become smaller and smaller or just bigger and bigger? What are different ways to make sense of data?

9. **CREATE AN ATMOSPHERE WHERE DIFFERENT THINKING STYLES ARE RESPECTED:** Show appreciation for the different thinking styles in your class. For example, "I see you like to take your time and think before you talk." "I see you really enjoy coming up with an answer right away."

10. **TAKE CHILDREN'S THINKING SERIOUSLY:** Instead of praising automatically, take the time to think of a serious response to a child's question, pattern, or verbalization of a thought.

11. **INVITE CHILDREN TO CONSTRUCT MULTIPLE REPRESENTATIONS:** Ask students to represent their ideas and problem-solving using different media (e.g., manipulatives, tables of data, graphs, pictures, journal-writing, equations). Constructing more than one type of representation helps children to become reflective, to compare the information that different representations yield, and to problem-solve about the important issue of representation itself.

The National Research Center on the Gifted and Talented
University of Connecticut
362 Fairfield Road, U-7
Storrs, CT 06269-2007
www.ucc.uconn.edu/~wwwgt

Production Assistants
Cathy Suroviak
Siamak Vahidi

Reviewers
Beverly Coleman
Denise de Souza Fleith
M. Katherine Gavin
E. Jean Gubbins

NRC
G/T

The National Research Center on the Gifted and Talented Research Teams

University of Connecticut

Dr. E. Jean Gubbins, Associate Director
University of Connecticut
School of Education, U-7
Storrs, CT 06269-2007
860-486-4676

Dr. Deborah E. Burns
Dr. Sally M. Reis
Dr. Karen L. Westberg

City University of New York, City College

Dr. Deborah L. Coates, Site Research Coordinator
City University of New York (CCNY)
138th and Convent Avenue
NAC 7/322
New York, NY 10031
212-650-5690

Stanford University

Dr. Shirley Brice Heath, Site Research Coordinator
Stanford University
Department of English
Building 50, Room 52U
Stanford, CA 94305
415-723-3316

Dr. Guadalupe Valdés

University of Virginia

Dr. Carolyn M. Callahan, Associate Director
Curry School of Education
University of Virginia
405 Emmet Street
Charlottesville, VA 22903
804-982-2849

Dr. Donna Y. Ford
Dr. Tonya Moon
Dr. Carol A. Tomlinson

Yale University

Dr. Robert J. Sternberg, Associate Director
Department of Psychology
Yale University
P.O. Box 208205
New Haven, CT 06520-8205
203-432-4632